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SOME BASIC RESPONSE RELATIONS
FOR REACTION-WHEEL ATTITUDE CONTROL

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FOR REACTION-WHEEL ATTITUDE CONTROL*

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ABSTRACT

In many space vehicles, attitude control is best accomplished with combination systems using reaction wheels for momentum exchange and storage, plus jets for periodic momentum expulsion. Design of the reaction-wheel control involves evaluating the time history of system response to disturbances, many of which are either sinusoidal or impulsive.

As an aid to such evaluation, this paper developes basic response relations--vehicle attitude, control torque, wheel motion, mechanical power, and energy consumption--for a vehicle subjected to both types of disturbance. Limiting values are calculated, assuming no standby losses. (The possibility of exchanging momentum with minimum energy loss is discussed.) The resulting normalized numerical relations are intended to serve as an order-of magnitude basis for preliminary design estimates and comparisons.

The response relations are derived first for a single-axis model. Then their applicability to three-axis design is discussed. A control system is postulated which decouples vehicle dynamics so that vehicle motions are exactly single axis. (Some advantages of such control are discussed in References (2) and (3).)
The resulting control-wheel motions may be complicated by gyroscopic coupling due to the spinning wheels. In control to a rotating reference extra power is consumed also because the spin momentum of the roll and yaw wheels must be passed back and forth from one to the other.

Control systems which merely damp the natural motions of stable, local-vertical satellites can be smaller and simpler and use less power but, of course, furnish less precise control. (They are generally useful only in nearly-circular orbits.)

*Based on work done for Systems Corporation of America, under contract with the U.S. Air Force, WADD Flight Control Laboratory, AF 33(616)-6674, and on research supported by a grant to Stanford University by the NASA. Part of the results in this paper are reported in Reference (1).

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NOMENCLATURE

A	Aerodynamic Surface. } Fig. 1.
a	Aerodynamic Moment arm. } Fig. 1.
E	Energy consumed by control system
e	Naperian base. Also, orbit eccentricity (Introduction only).
$\overline{H}$	Vector angular momentum of a system.
h_1, h_2, h_3	Initial angular momentum of reaction wheels.
I	Vehicle moment of inertia.
J	Reaction wheel moment of inertia.
j	Unit vector along imaginary axis of s-plane.
K	Control system gain (see Fig. 3).
k	Proportionality constant for attitude-dependent torques.
$\mathbf{L}$	External torque on vehicle.
L	Laplace transform of $\mathbf{L}$.
$\overline{M}$	External torque on a system.
n	Constant angular velocity of a local-vertical reference in a circular orbit.
P	Power required by control system.
p	Symbol representing the operation of differentiation: $p \triangleq \frac{d}{dt}$.
s	Parameter used in Laplace transform.
$\mathcal{T}_c$	Torque exerted on vehicle by control device.
T_c	Laplace transform of $\mathcal{T}_c$.
t	Time.
$\overline{1}, \overline{2}, \overline{3}$	Unit vectors along principal axes of vehicle.
$\overline{1}_r, \overline{2}_r, \overline{3}_r$	Unit vectors along reference axes.
$\delta(t)$	Unit impulse.

τ	Time constant of controlled-vehicle response.
θ	Vehicle attitude deviation from the reference direction.
Θ	Laplace transform of θ.
σ	A control system parameter (Fig. 3).
ψ	Phase angle.
Ω	Reaction wheel speed.
Ω_o	Initial reaction-wheel speed.
$\mathcal{\Omega}$	Laplace transform of Ω.
ω	Angular velocity of reference frame. (When ω is constant it equals n.)
ω_f	Forcing frequency.

Subscripts

1,2,3	Component of a quantity along vehicle axis 1, 2, or 3.
j	1, 2, or 3.
v	Pertaining to vehicle.
max.	Maximum value.
r	Pertaining to reference.
d	Pertaining to damping system of last section.
p,r,y	Pitch, roll, yaw.

INTRODUCTION

Combination Systems.

For common space vehicle attitude control systems a combination of both momentum expulsion devices (e.g., jets) and momentum storage devices (e.g., reaction wheels or gyros) are often used as actuators. The jets and momentum storage devices complement one another, the storage devices countering cyclical torques on the vehicle without loss, and the jets countering long-term secular torques by periodically expelling momentum from the storage system as the storage devices near spin saturation.

Dynamic Considerations in Design.

In combination systems the jets usually operate as stand-by devices. Typically, the momentum stored in the storage elements is monitored and the appropriate jets are fired when stored momentum reaches some percentage of total storage capacity. In such systems design requirements for the jet system are evident: (1) the total stored gas must represent a momentum capacity somewhat larger than the total secular impulse anticipated over the life of the vehicle; (2) the maximum torque capacity of the jets must be somewhat larger than the maximum torque anticipated on the vehicle--for example, the misalignment torque during on-orbit maneuver rocket firing; and (3) the impulse of the jet system (moment times time) should be controllable to within a few percent of the total storage capacity of the momentum storage system (so that momentum can be expelled from the storage system with precision). In all of these requirements the dynamic behavior of the jet-controlled vehicle is of secondary importance compared with precision control of the jet thrust level and the thrust time.

The momentum storage elements must be capable of storing, without saturation, the largest cyclical impulses anticipated plus the secular momentum change between momentum expulsions.

But, in addition, the momentum storage section draws also the assignment of furnishing the specified attitude-control precision and speed of response. Moreover, it must consume minimum power and energy in the process. Dynamic behavior is therefore of prime importance here.

The present paper is concerned with this latter aspect of attitude control system design--the dynamic behavior and attendant mechanical power requirements of reaction-system-controlled space vehicles. In the first section of the paper response relations are developed for a single-axis model of a space vehicle controlled by a reaction wheel. In the two succeeding sections three-axis behavior of a vehicle is studied for control using a set of reaction wheels.

In subsequent papers the behavior of gyro-controlled systems will also be studied.

Typical Disturbances.

Two forms of disturbance torque--an impulse and a sinusoid--have been chosen for the study of dynamic response. Sinusoidal disturbances commonly predominate for a vehicle in a circular orbit about a planet. For example, on a space vehicle whose reference is inertially fixed (such as the orbiting astronomical observatory) the aerodynamic torque will vary sinusoidally at orbital frequency and the gravity-gradient torque at twice orbital frequency. (At the OAO altitude gravity-gradient torque will predominate.)

For a vehicle controlled to the local vertical aerodynamic torque may also be sinusoidal (at orbital frequency) if the orbit is slightly eccentric, as shown in Fig. 1*.

Figure 1 also shows that if the orbit eccentricity is somewhat larger, aerodynamic torque on the vehicle is essentially an impulse. The reason is, of course, that atmospheric density falls away so rapidly with altitude that in an eccentric orbit the vehicle spends only a small portion of its orbit period in the effective atmosphere.

For intermediate values of eccentricity the aerodynamic torque may be represented by a Fourier series, involving the first few multiples of orbit frequency, so that, again, sinusoidal response is of interest. (The Fourier series approach has been employed, for example, by Schrello in Reference (4).) Sources and magnitudes of disturbing torques have been discussed in a number of papers (c.f. References (4), (5), (6)). Certain of their effects will be discussed further in Reference (3).

Other instances of impulsive disturbances may occur (e.g., meteorite impact). More generally, impulse response furnishes a good starting point for analysis of response to random disturbances. In the present paper sinusoidal and impulse disturbances of typical size will be used to illustrate the form and order of magnitude of salient response relations.

SINGLE AXIS CONTROL**

Figure 2 offers a single-axis model of a space vehicle with a reaction wheel for attitude control.*** It is shown in the next section that this model can represent the actual situation quite well in many situations of interest; namely, when the attitude stays near a reference which is inertially non-rotating, wheel speeds are slow, wheels are accurately aligned to vehicle principal axes, and the vehicle is

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*Calculations for Fig. 1 are by Roger Bourke, Stanford University.

**Parts of this section are repeated in Reference (2).

***The variables are attitude deviation, θ, and wheel speed, Ω. L is an external disturbing torque, and T_c is the internal control torque between the wheel and the vehicle.

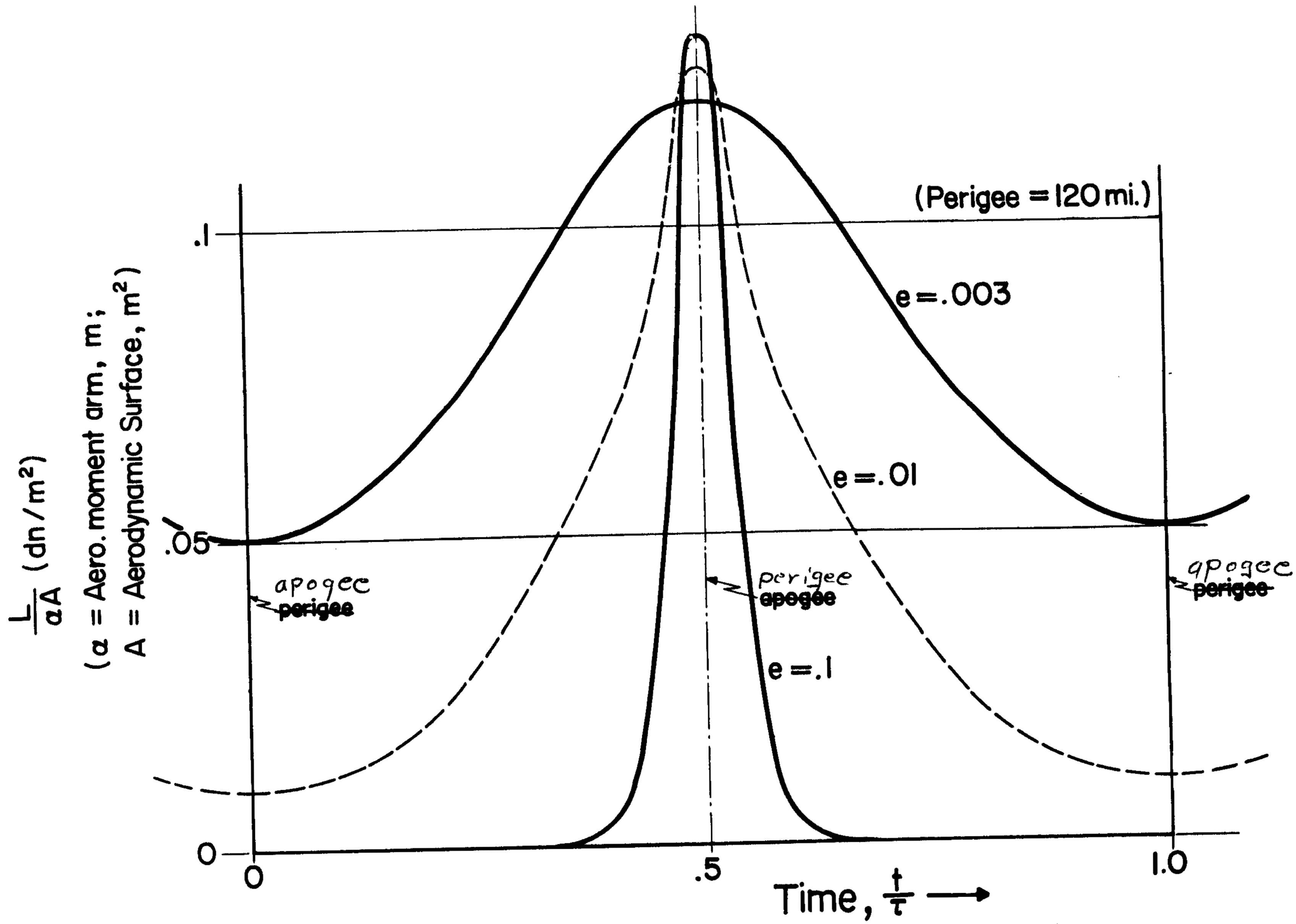

FIG. 1 AERODYNAMIC TORQUE ON A HYPOTHETICAL LOCAL LEVEL SATELLITE

symmetrical with respect to external torques. In such situations dynamic behavior about each axis is decoupled and may be treated independently. Moreover, as will be shown, single-axis results furnish a most useful reference for three-axis studies.

In the present simple study it is assumed also that control will be "stiff" compared with physical restoring torques, such as gravity-gradient or aerodynamic θ-dependent torques, and these torques will be omitted at this point. Their effect is discussed at the end of the present section, and also in the next section.

Dynamic Equations*

Three equations of motion can be written for the system of Fig. 2, any two of which constitute an independent set in the variables θ and Ω. Newton's Second Law is applied, in turn, to the systems of Fig. 2(a) (vehicle plus wheel), 2(b) (vehicle only), and 2(c) (wheel only).

From Fig. 2(a), for the vehicle plus wheel:

$$(I + J)\ddot{\theta} + J\dot{\Omega} = L \quad ,$$

or, after Laplace transformation:

$$(I + J)s^2\theta + Js\Omega = L \tag{1a}$$

From Fig. 1(b), for the vehicle only:

$$Is^2\theta = L + T_c \tag{1b}$$

From Fig. 1(c), for the wheel only:

$$Js(\Omega + s\theta) = -T_c \tag{1c}$$

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*The equations of motion for a much more general situation have been derived by Roberson. cf., Reference (7)

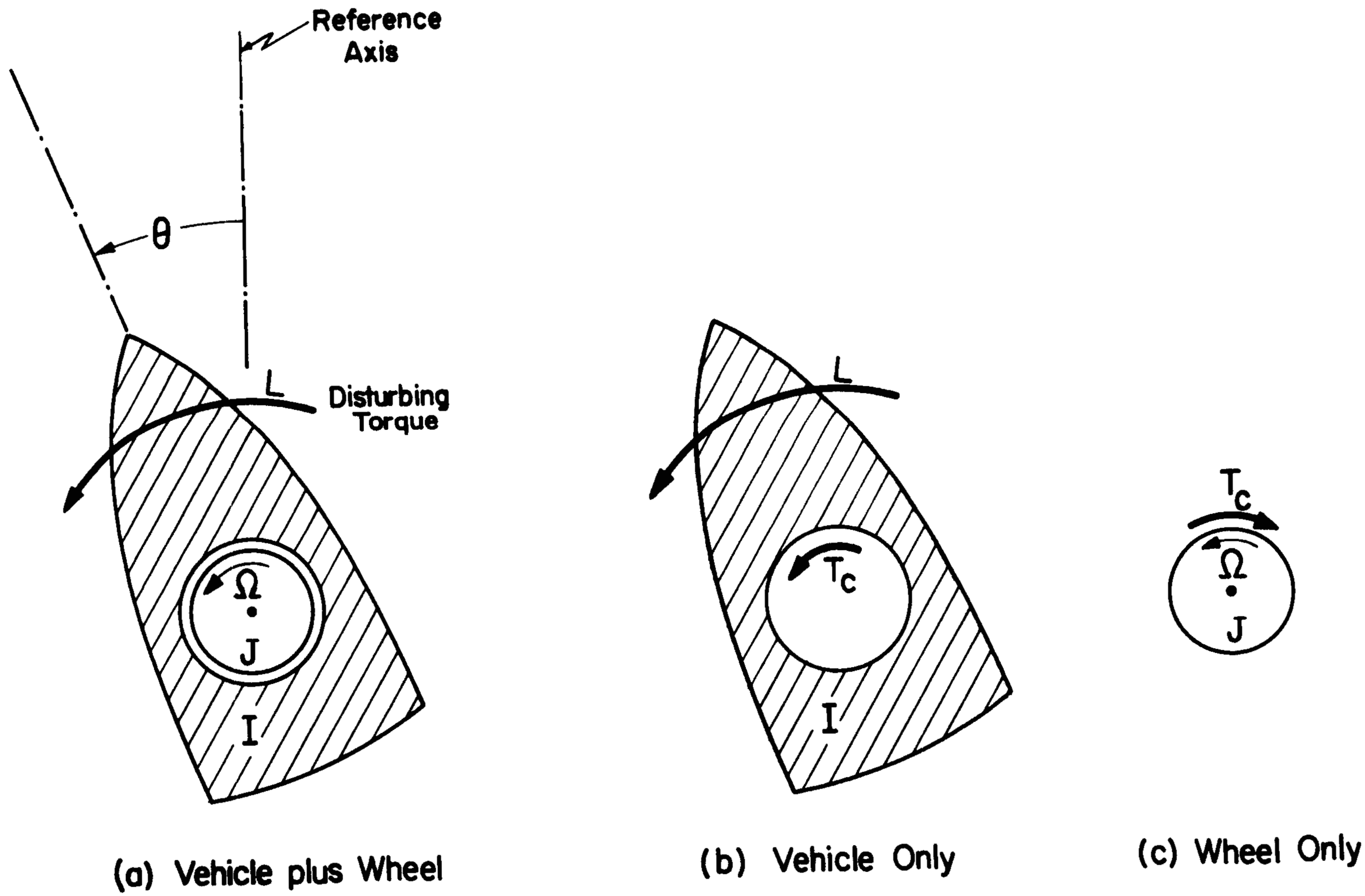


FIG. 2 SINGLE-AXIS MODEL OF VEHICLE WITH REACTION WHEEL

In Eq. (1) T_c is the control torque, applied by the control system between the wheel and the vehicle. Ω is the angular velocity of the wheel relative to the vehicle. Note that friction torque and back e.m.f. between wheel and vehicle are assumed zero in this study.

Control System.

A simple control system is shown in Fig. 3 for controlling the attitude of the vehicle in Fig. 2. An attitude sensor (e.g., a star-sight system) is assumed to report θ .

The upper part of the block diagram in Fig. 3 is taken directly from Eq. (1b). To this is added a simple control loop employing proportional plus derivative control. For simplicity the control gain is set so that the system characteristic has two equal real poles, as given by the root locus picture in Fig. 3b. The gain required for this is:

$$K = \frac{I}{\tau^2}$$

in which τ is the final time constant of the system.

With this gain setting, the overall system transfer function is

$$\frac{\theta}{L} = \frac{\tau^2}{I} \frac{1}{(\tau s + 1)^2} \qquad (2)$$

Response to Initial Attitude Error.

The response of the system of Fig. 3 to an initial value of θ is:

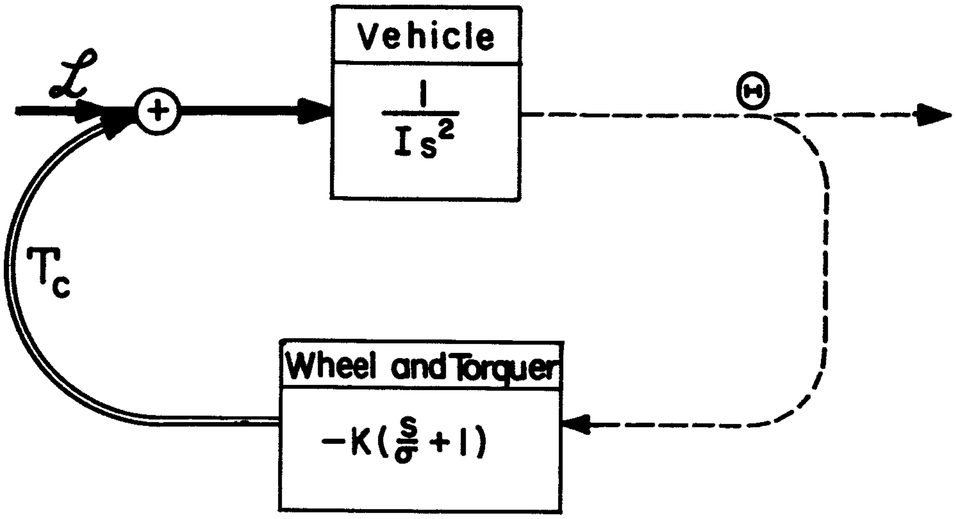
$$\theta = \theta(o)\left(1 + \frac{t}{\tau}\right)e^{-\frac{t}{\tau}} \qquad (3)$$

Response to initial θ is plotted to a linear scale in Fig. 4a, and to a semi-log scale in Fig. 4b. Choice of τ for adequately fast recovery from initial errors can be made from Fig. 4. For example, if the initial misalignment must be reduced by a factor of 100 in 1500 sec. then, from Fig. 4b, 6.6τ must equal 1500, and the required system time constant is $\tau < 230$ sec.

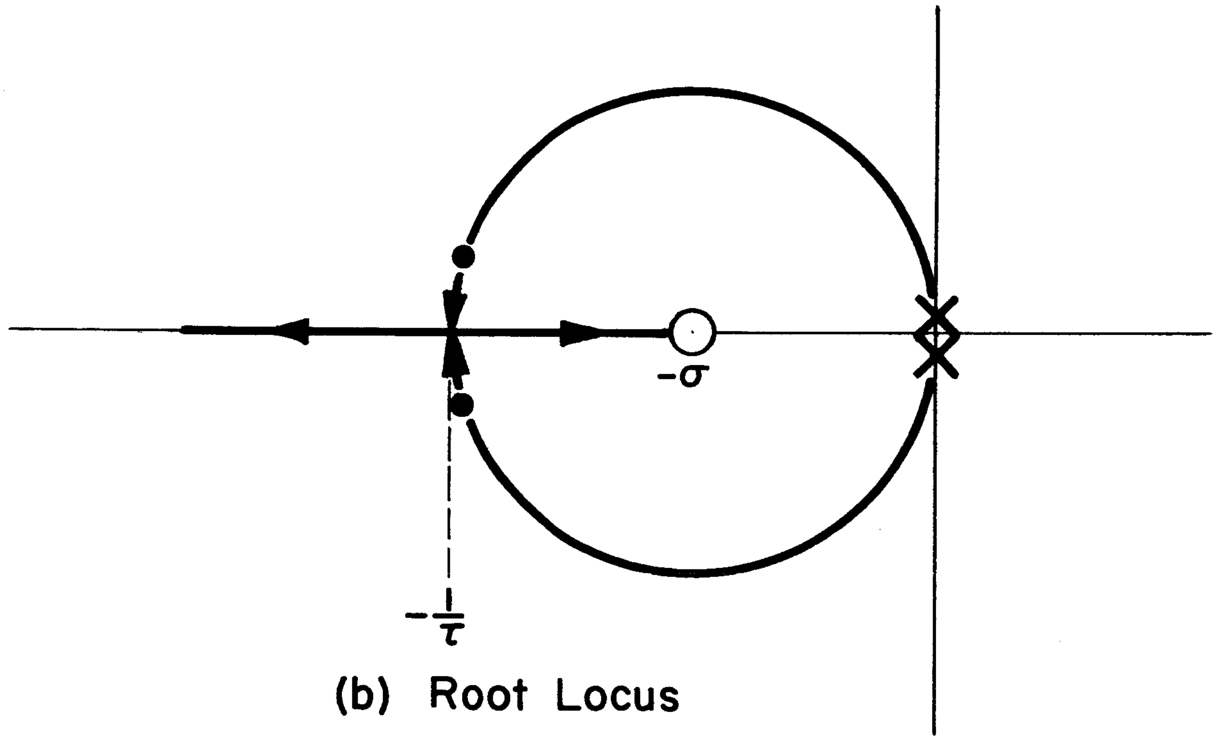
Response Relations for Impulse Disturbance.

(a.) Attitude Excursion

If L is an impulse of magnitude $\ell = L\,\Delta t$, as shown in Fig. 5a, then its Laplace transform is $L(s) = \ell$, and the system response is given by



(a) System Arrangement



(b) Root Locus

FIG. 3 ATTITUDE CONTROL SYSTEM

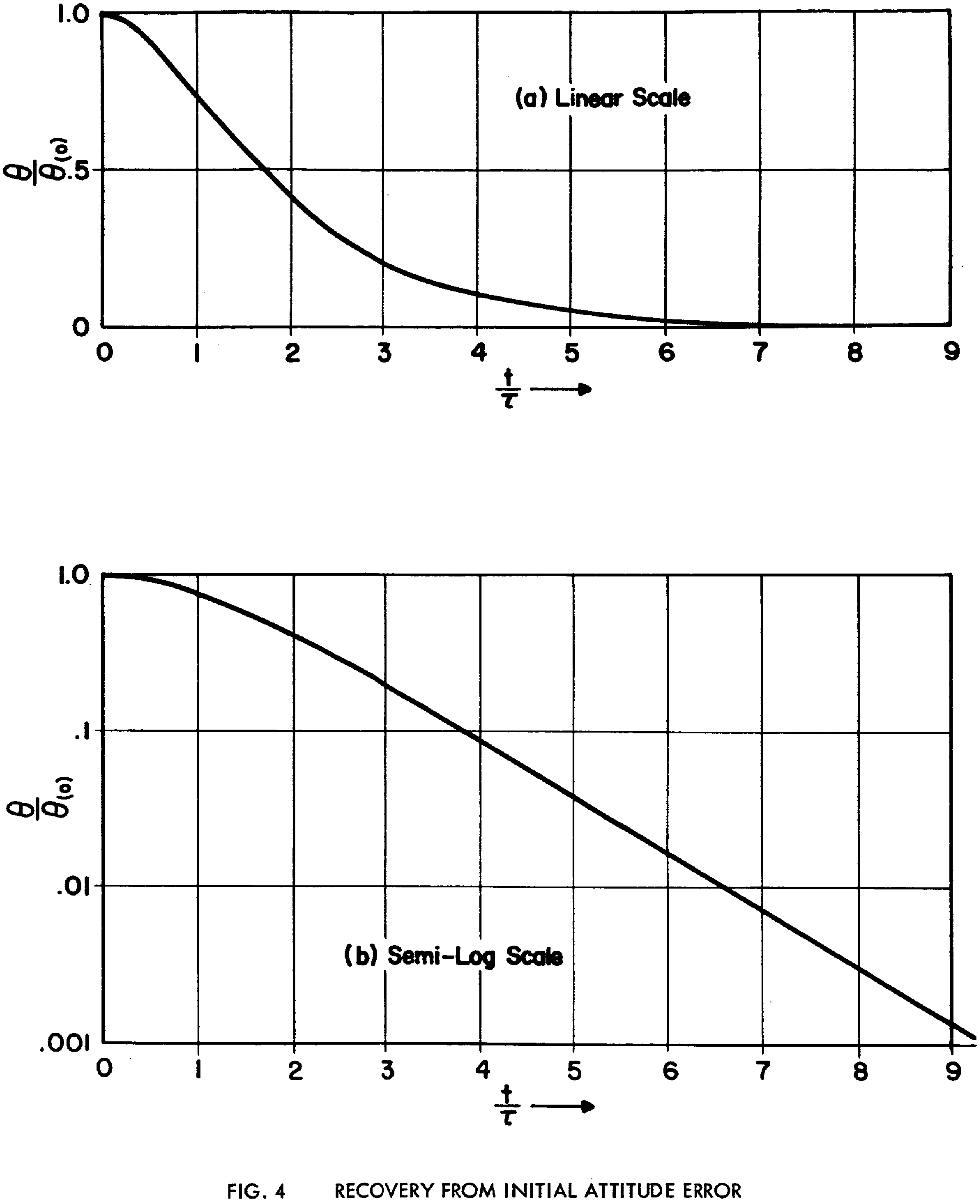


FIG. 4 RECOVERY FROM INITIAL ATTITUDE ERROR

$$\theta(t) = \frac{\ell\tau}{I}\,\frac{t}{\tau}\,e^{-\frac{t}{\tau}} \tag{4}$$

This relation is plotted in Fig. 5b.

The maximum deviation of the vehicle occurs at $t = \tau$, and has the value:

$$\theta_{max.} = \frac{\ell\tau}{I}\,\frac{1}{e} \tag{5}$$

For purposes of numerical calculation, for specific design problems, it might be convenient to have available a set of response curves for various values of τ. Such a set is given in Fig. 6, from which not only can the maximum vehicle deviation (from its reference attitude) be read, but also the length of time the deviation will exceed any specified value.

For example, suppose the appropriate control-system time constant is to be chosen for a vehicle of moment of inertia $2\diagdown^{10}$ gm cm^{2}* whose peak attitude excursion, under an impulsive disturbance of $4\diagdown^{6}$ dn cm sec, is to be held under 10 mr, and--further--whose misalignment may exceed 1mr for no longer than 200 sec after the same disturbance. The first specification gives $\dfrac{\theta_{max}}{\ell/I} = \dfrac{1\diagdown^{-2}}{(4\diagdown^{6})/2\diagdown^{10}} = 50$, for which, from Fig. 6, τ must be made less than about 140 sec. But the second specification requires that $\dfrac{\theta}{\ell/I}$ be larger than $\dfrac{1\diagdown^{-3}}{4\diagdown^{6}/2\diagdown^{10}} = 5$ for no longer than 200 sec which, from Fig. 6, means τ must be less than about 50 sec.

(b.) <u>Control Torque Required</u>.

From Fig. 3 and Eq.(2) wheel control torque is to be related to vehicle disturbing torque by the transfer function:

$$T_c = -\frac{I}{\tau^2}(2\tau s + 1)\Theta = -\frac{(2\tau s + 1)}{(\tau s + 1)^2}\,L \tag{6}$$

from which, for L an impulse of magnitude ℓ, the time response is:

$$T_c(t) = -\frac{2\ell}{\tau}\left(1 - \frac{t}{2\tau}\right)e^{-\frac{t}{\tau}} \tag{7}$$

This response is plotted in Figure 7. Note that the control torque has its peak magnitude at $t = 0$, the instant at which the disturbing impulse occurs. Its value is:

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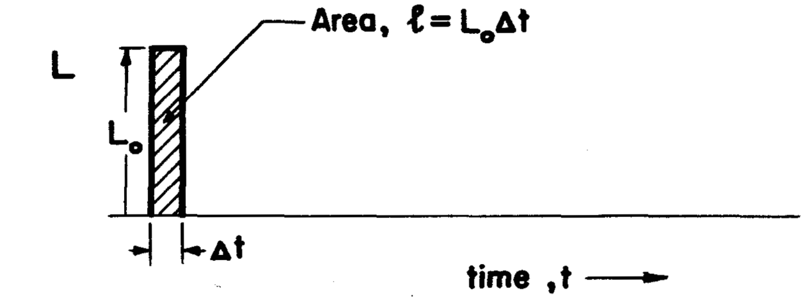
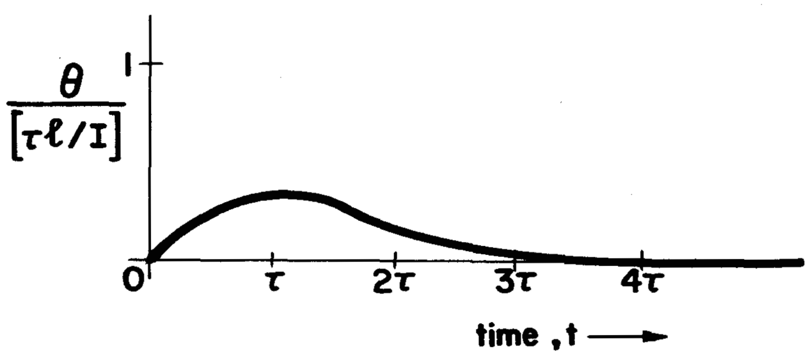
*The symbol $\diagdown^{n}$ is used to abbreviate "x10^{n}." For example, $2\diagdown^{7}$ means $2\text{x}10^{7}$.



(a) Impulsive Disturbing Torque



(b) Vehicle Response

FIG. 5. VEHICLE RESPONSE TO IMPULSIVE DISTURBANCE.

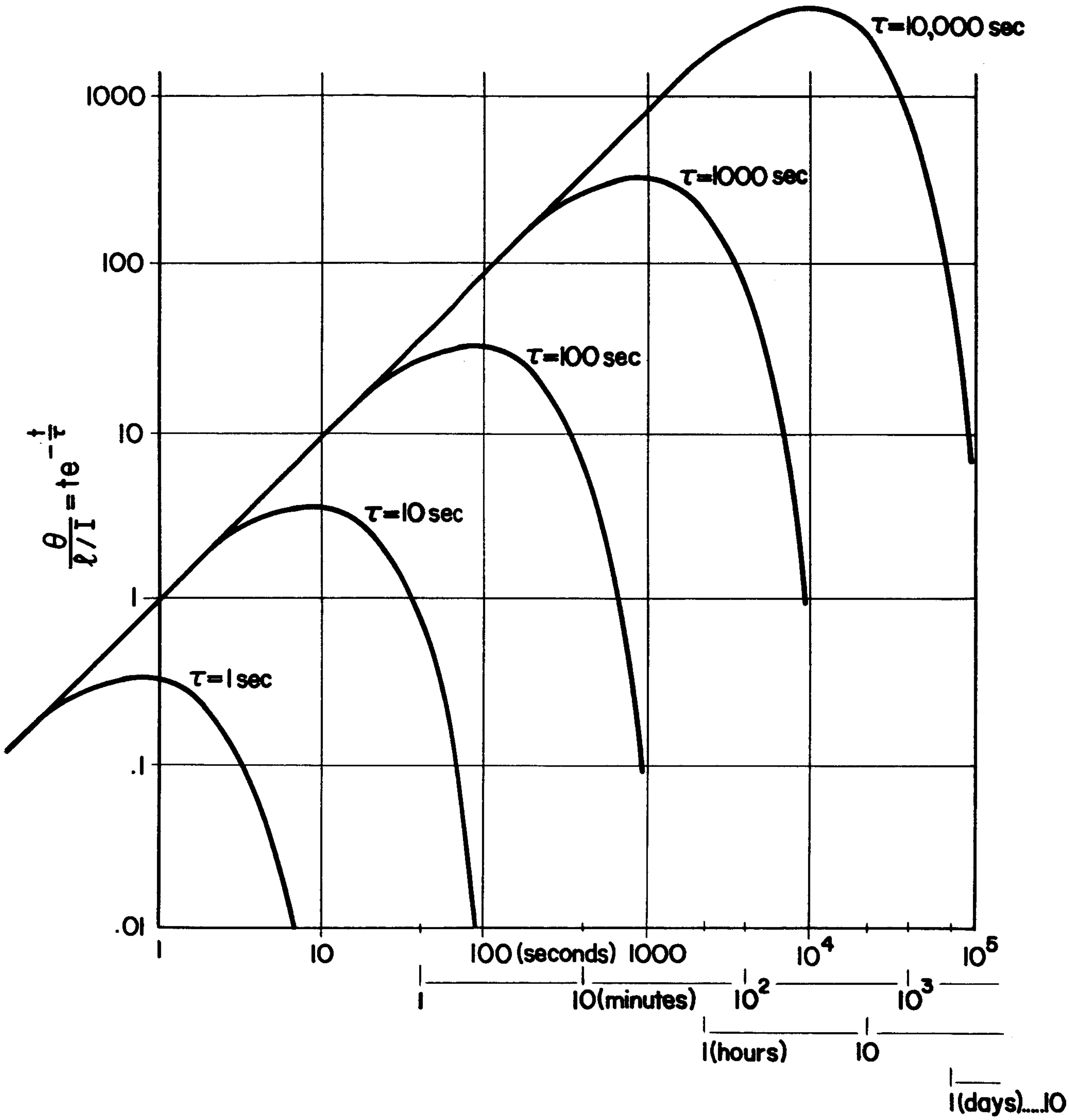


FIG. 6 DESIGN RELATIONS

$$T_{c_{max}} = -\frac{2\ell}{\tau} \tag{7a}$$

A more direct way to derive the desired control is to solve Eq.(1a) for Ω or (1b) for T_c, after substituting for L from the desired-response function, (2). The result is*:

$$T_c = -Js\Omega = +Is^2\Theta - \frac{I}{\tau^2}(\tau s+1)^2\Theta = -\frac{I}{\tau^2}(2\tau s+1)\Theta \quad ,$$

or

$$\Omega = \frac{I}{J\tau^2}\frac{2\tau s+1}{s}\Theta \quad , \tag{8a}$$

$$T_c = -\frac{I}{\tau^2}(2\tau s+1)\Theta \quad ,$$

the last expression being the same as (6).

(c.) Wheel Motion Required.

From Eqs. (1c) and (6) or (2) and (8a) the relation between the vehicle disturbing torque and control wheel motion is given by:

$$\Omega = \frac{1}{Js}\frac{(2\tau s+1)}{(\tau s+1)^2} L \tag{8b}$$

Again, if the disturbance is an impulse, the time response is:

$$\Omega(t) = \frac{\ell}{J}\left[1 + \left(\frac{t}{\tau} - 1\right)e^{-\frac{t}{\tau}}\right] + \Omega_o \quad , \tag{9}$$

which is plotted in Fig. 8 for initial speed, $\Omega_o = 0$. The wheel, of course, rotates in a positive direction (by the convention of Fig. 2), so that it, rather than the vehicle, will have the momentum change required by the application of torque impulse ℓ to the overall system.

(d.) Power and Energy Required.

The minimum value of power required by the control system will be that needed to accelerate the reaction wheel, assuming no friction, and assuming that no power is dissipated in electrical losses.** This power is simply the product of control torque and wheel speed relative to the vehicle. (This is a dot product--that is, it involves the sign of both quantities).

*Term $Js^2\Theta$ is dropped in (1a) on the grounds that $\frac{J}{I} << 1$.

**Only this limiting value of mechanical power is considered in the present paper.

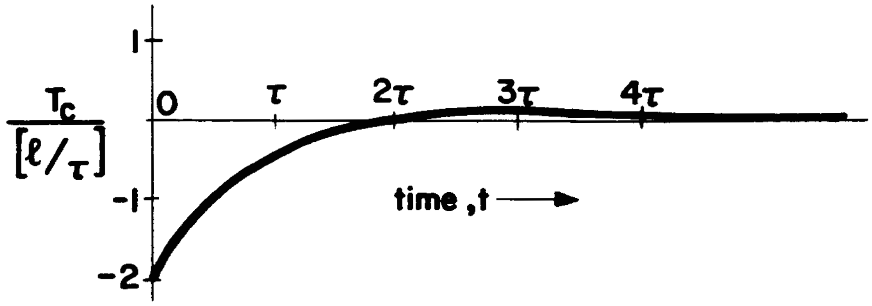


FIG. 7 CONTROL TORQUE RESPONSE TO IMPULSIVE DISTURBANCE

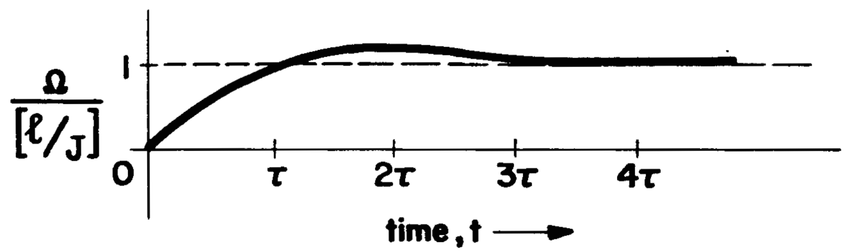


FIG. 8 WHEEL SPEED RESPONSE TO IMPULSIVE DISTURBANCE

Thus, for an impulsive disturbance, power is given by the product of $-T_c(t)$ from Eq.(7) and $\Omega(t)$ from Eq.(9). (The minus sign on T_c is necessary because Eq.(7) gives torque on the vehicle while the present calculation is concerned with torque on the wheel. Note that the sign convention in Fig. 2 considers torque on the wheel and wheel velocity positive in opposite directions.) For the special case that $\Omega_o = 0$ the power is:

$$P(t) = \frac{2\ell^2}{J_\tau}\left[\left(1 - \frac{t}{2\tau}\right) e^{-\frac{t}{\tau}} - \left(1 - \frac{t}{2\tau}\right) e^{-2\frac{t}{\tau}}\right] \tag{10a}$$

$P(t)$ is plotted in Fig. 9a. Maximum power is required at about $t = \frac{\tau}{2}$ and has the value:

$$P_{max} = .66 \frac{\ell^2}{\tau J}$$

Note that the power, as plotted, actually becomes negative after time $t = 2\tau$, when the wheel is being decelerated as it approaches its final speed. This result is based on the premise of recovering energy from the spinning wheels when they are to be slowed down, and storing the energy (for example, in another wheel spinning in the opposite direction). The feasibility of such energy conservation is discussed in Appendix B. If energy cannot be recovered then, of course, the magnitude ~~only~~ the positive power should be read in Fig. 9. (Slowdown torque would presumably be applied by a brake.)

The total energy consumed is found by integrating Eq.(10a) or finding the area under the curve in Fig. 9a. The value obtained is:

$$E = .5 \frac{\ell^2}{J} \approx P_{max}\,\tau \tag{11a}$$

assuming, again, that negative values in Fig. 9 represent recoverable power. Energy consumed would, of course, be slightly larger if no recovery were assumed.

If a wheel is spinning with some high initial speed, Ω_o, when the vehicle is disturbed (e.g., by impulse, ℓ), then a much higher power level is required to accelerate the wheel. In this case the power is given by an amended version of (10) (obtained by multiplying (7) by (9)):

$$P = \frac{2\ell^2}{J\tau}\left[\left(1 - \frac{t}{2\tau}\right) e^{-\frac{t}{\tau}} - \left(1 - \frac{t}{2\tau}\right)\left(1 - \frac{t}{\tau}\right) e^{-2\frac{t}{\tau}}\right] + \frac{2\ell\Omega_o}{\tau}\left(1 - \frac{t}{2\tau}\right) e^{-\frac{t}{\tau}} \tag{10b}$$

in which the last term may be much larger. When this is so the power consumed will look like Fig. 9b, and the energy will also be much larger:

$$E = \ell\Omega_o \tag{11b}$$

That is, the torquer must expend much additional energy to chase the

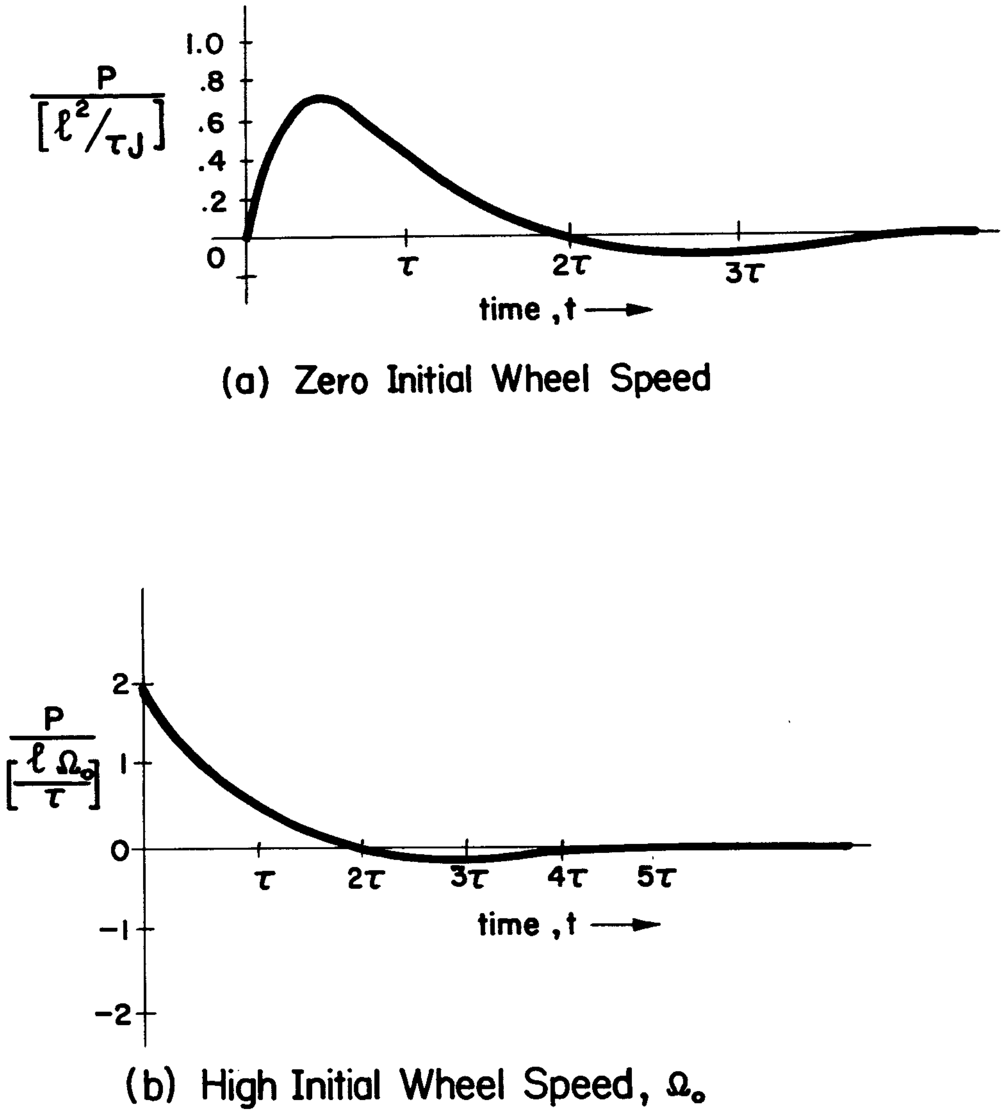


FIG. 9 POWER CONSUMED IN RESPONDING TO IMPULSE

spinning sheel. This energy is larger than the zero-initial-wheel-speed energy by the ratio

$$\frac{\ell\Omega_o}{.5\ell^2/J} = 2\,\frac{J\Omega_o}{\ell} = 2\,\frac{h}{\ell}$$

in which h is initial wheel momentum.

<u>Response Relations for Sinusoidal Disturbance</u>.

Suppose that the system just described is disturbed by a sinusoidal external torque:

$$L = L_o \cos \omega_f t \tag{12}$$

Then, from (2) (using, for example, the steady-state portion of the Laplace transform):

$$\theta = \theta_{max} \cos (\omega_f t + \psi_\theta) \quad ,$$

in which

$$\theta_{max} = \frac{L_o \tau^2}{I} \left| \left[\frac{1}{(\tau s + 1)^2} \right]_{s=j\omega_f} \right| \tag{13}$$

and

$$\psi_\theta = \angle \left[\frac{1}{(\tau s + 1)^2} \right]_{s=j\omega_f}$$

Usually ω_f will be orbital frequency, while τ will be shorter than $1/2\pi$ times the orbital period, so that $\tau\omega_f < 1$, and the relations are simpler:

$$\text{For } \tau\omega_f < 1; \quad \theta = \frac{L_o \tau^2}{I} \cos \omega_f t \tag{14}$$

The corresponding values of control torque and wheel speed are, from (6) and (8):

$$\text{For } \tau\omega_f < 1 \begin{cases} T_c = - L_o \cos \omega_f t & (15) \\ \Omega = \dfrac{L_o}{J\omega_f} \sin \omega_f t & (16) \end{cases}$$

Then the power required is:

$$P = -T_c \Omega = \frac{L_o^2}{J\omega_f} \; \frac{\sin 2\omega_f t}{2} \tag{17}$$

which is plotted in Fig. 10.

It is important to note that the power has no secular component, which might have resulted from rectification when two sine waves were multiplied together. There will never be rectification (for the ideal, no-damping model assumed here) because Ω is always the exact integral of T_c: they will always be 90^o out of phase.

If energy can be recovered as the wheels are driven, then the net energy per cycle will be zero. But if, on the other hand, a drive-then-brake method is used, then the energy consumed will be the shaded area in Fig. 10 (braking involves no power):

$$E = \frac{L_o^2}{J\omega_f^2} \quad \text{per cycle} = 2\,\frac{P_{max.}}{\omega_f} \text{ per cycle} \tag{18}$$

As an example, suppose a vehicle having moment of inertia $I = 2\backslash^{10}$ gm cm^2 is controlled to an inertial reference with wheels of inertia $J = 2\backslash^{4}$ gm cm^2 and with a time constant of 100 sec. Suppose the vehicle is in a slightly eccentric orbit such that the sum of aerodynamic and gravity-gradient torque is sinusoidal, at orbit frequency, with peak magnitude $1\backslash^{4}$ dn cm. Then, from (14), (15), (16), and (17), the peak values of vehicle attitude excursion, control torque, wheel speed, and power required will be, respectively,

$$\theta_{max} = \frac{L_o \tau^2}{I} = \frac{(1\backslash^{4})(1\backslash^{2})^2}{2\backslash^{10}} = .005 \text{ rad.}$$

$$T_{c_{max}} = L_o = 1\backslash^{4} \text{ dn cm}$$

$$\Omega_{max} = \frac{L_o}{J\omega_f} = \frac{1\backslash^{4}}{(2\backslash^{4})\;(1\backslash^{-3})} = 500 \text{ rad/sec.}$$

$$P_{max} = \frac{1}{2} T_{c_{max}} \Omega_{max} = \frac{1}{2}\,\frac{(1\backslash^{4})\;(5\backslash^{2})}{1\backslash^{7}} = 25 \text{ watts}$$

(The factor $1\backslash^{7}$ converts dn cm/sec. to watts.)

Without energy recovery, the energy used per cycle, by (18), is:

$$E = \frac{L_o^2}{J\omega_f^2} = \frac{(1\backslash^{4})^2}{(2\backslash^{4})\;(1\backslash^{-3})^2\;1\backslash^{7}} = 500 \text{ watt-sec.}$$

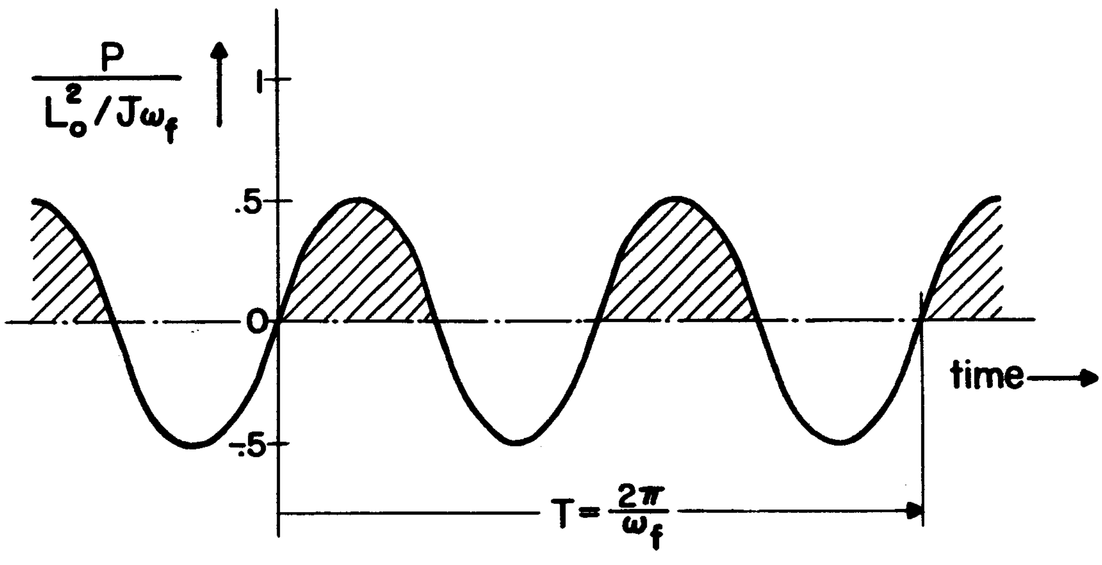


FIG. 10 POWER CONSUMED DUE TO SINUSOIDAL DISTURBANCE

In cases where $\tau\omega_f$ is larger than 1 (i.e., the system time constant can be made longer than an orbital period; or a high-frequency disturbance is involved) the above quantities may be attenuated, as indicated by the functions of s in (2), (5) and (8). This is discussed further in the last section of this paper.

Effect of Physical Restoring Torques.

Sometimes, if the system time constant is to be very slow, physical θ-dependent torques--particularly gravity-gradient or aerodynamic torques--may be important factors in the dynamic characteristices of the system.

In such cases the external torque on the vehicle consists not only of the independent component, L , in Eq.(1a), but also of a θ-dependent component. In the case of a local-vertical satellite, and for the very small θ's we are considering, this torque may be written as a linear function, $-k\theta$. Then Eq.(1a) becomes

$$Is^2\theta + Js\Omega = L - k\theta$$

or

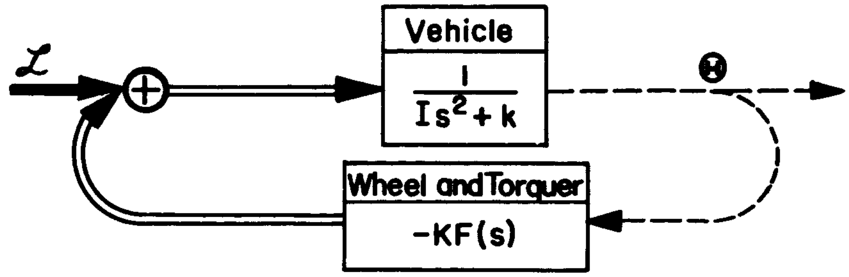
$$(Is^2 + k)\theta + Js\Omega = L \tag{19}$$

In applications where positive, precise control is indicated, the presence of such additional torques may be taken care of by having the control system cancel them out. That is, the term $k\theta$ in (19) would be countered by amending the control torque equation, (8a),as follows:

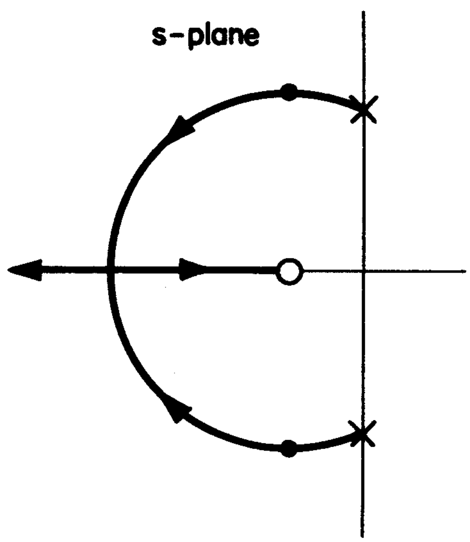
$$Js\Omega = \left[\frac{I}{\tau^2} (2\tau s + 1) - k \right] \theta \tag{20}$$

With this control the attitude response of the vehicle will be exactly as given by (2), as can be seen by substitution of (20) into (19). Control torque and wheel speed would, of course, be altered by the additional control term $k\theta$. As a practical matter, however, the magnitudes of physical torques will be small, compared to control torques involved in precise attitude control, so that only small percentage changes in T_c and Ω would be involved.

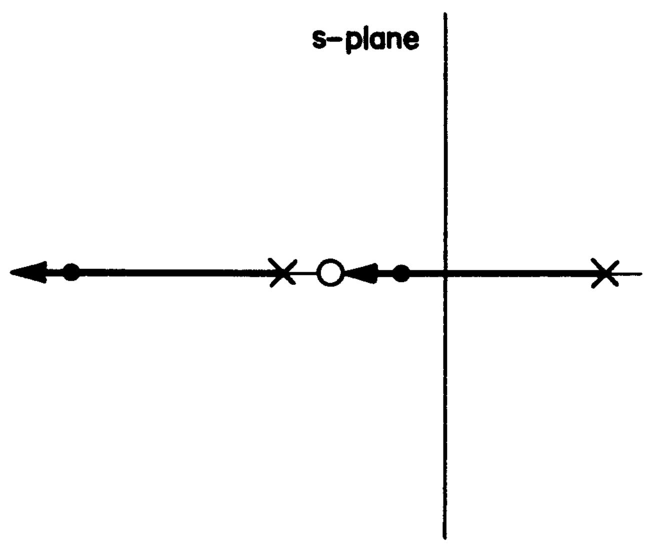
Occasionally only light attitude stability is required and only approximate attitude angle must be maintained. In such cases it may be possible to use physical restoring torques to supply the main stabilizing torque, with the attitude control system functioning only to supply damping. This situation is depicted in Fig. 11a, which is similar to Fig. 3, except that the vehicle dynamics now include the physical restoring torque (e.g., due to gravity gradient). The corresponding root locus picture for the case that k is positive (stable restoring torque), is shown in Fig. 11b. Without the control system, the vehicle would oscillate indefinitely at its own natural frequency. The control system supplies damping, so that natural motions of the vehicle are damped out in time.



(a) Control System



(b) k Stabilizing



(c) k Destabilizing

FIG. 11 CONTROL WITH PHYSICAL RESTORING TORQUE

The use of passive stabilization (with control system damping) is probably confined to those applications in which the orbit can be counted upon to be nearly circular (e.g., eccentricity < .001). Otherwise, as some studies have shown, (c.f. References (8) and (9)), the gravity gradient torque may actually cause dynamic instability for some initial conditions of the vehicle.

THREE-AXIS CONTROL

Dynamic Equations.

The equations of motion for a space vehicle with internal moving parts have been written elsewhere for quite general circumstances. (c.f. Reference (7).) However, it will be convenient to rederive the dynamic equations briefly for the special case of interest here.

Consider the space vehicle shown in Fig. 12 with principal axes $\bar{1}$, $\bar{2}$, and $\bar{3}$ fixed in the vehicle, and assume the special case that the three control wheels have their spin axes aligned exactly along the principal axes of the vehicle, as shown. The attitude of the vehicle is to be controlled to the reference axes $\bar{1}_r$, $\bar{2}_r$, and $\bar{3}_r$, which are also an orthogonal set. (In general, the reference axes may be inertially fixed or may rotate in some specified manner). Let the orientation of the vehicle with respect to the reference axes be defined by angles θ_1, θ_2, θ_3, as shown. (It is assumed that θ's are always small.)

Next, assume further that the reference system is either an inertial one, or is orbit-oriented to the local vertical and the orbit axis. In the latter case, to simplify the equations, we let the $\bar{2}_r$ axis coincide with the orbit axis and the $\bar{3}_r$ axis coincide with the vertical, so that the reference system has an angular velocity, say ω, about the $\bar{2}_r$ axis only. (Except for a perfectly circular orbit, ω will vary with time.) The equations written for this reference system can be quickly specialized to the inertial reference case by letting ω equal zero.

The system of Fig. 12 has three degrees of freedom to locate its mass center, plus six more to define motion with respect to the mass center (three for the rigid vehicle, plus one each for the wheels relative to the vehicle). It can be shown that, for situations of interest, motions about the mass center will have a negligible effect on motion of the mass center. For present purposes, therefore, we can consider variables defining motion of the mass center as independent, and write just six dynamic equations for motion about the mass center. We shall write them using the three vehicle attitude angles, θ, plus the three wheel speeds, Ω as the dependent variables. (In each case we write the well known extension of Newton's Second Law for motion of a system of rigid bodies about their mass center.)

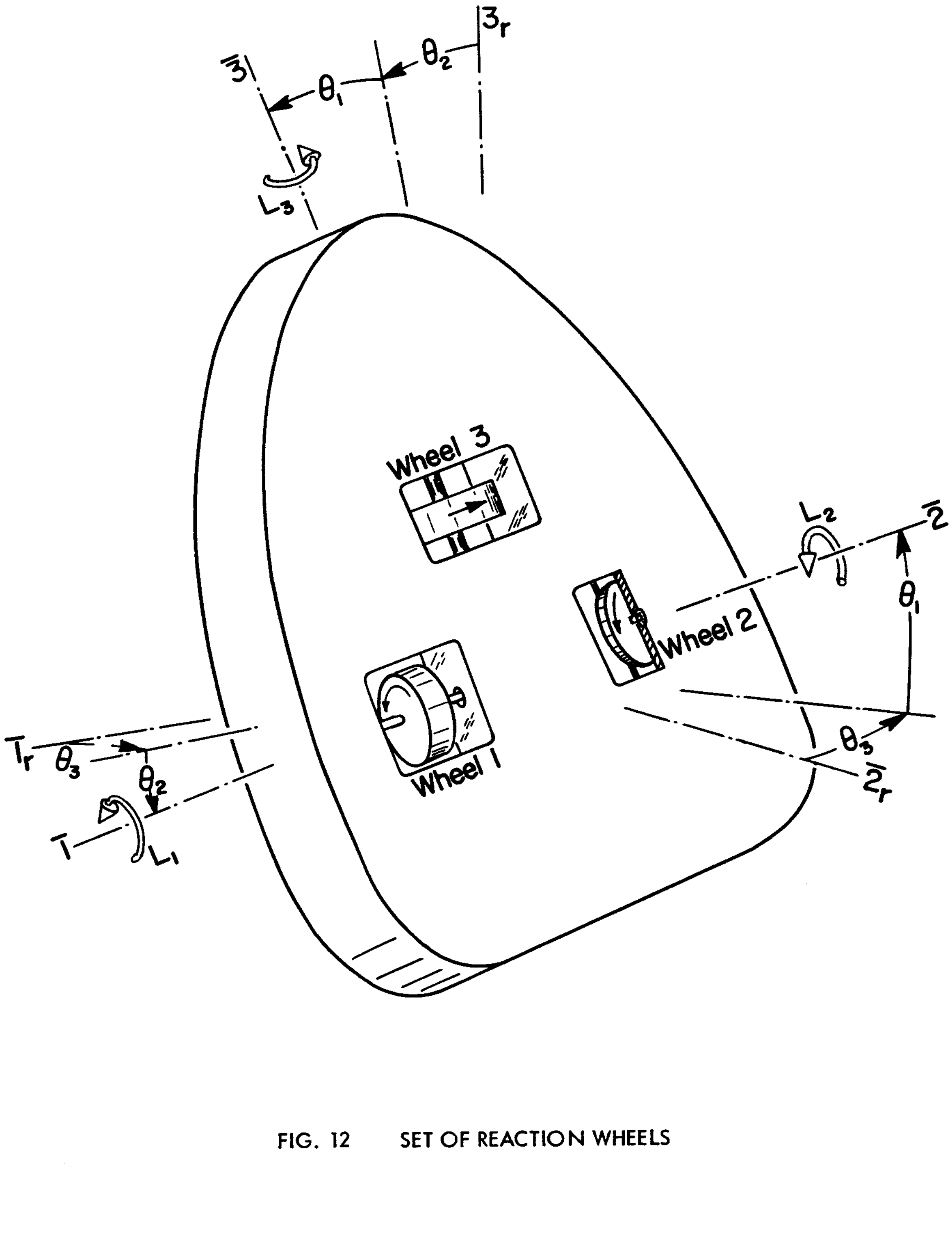


FIG. 12 SET OF REACTION WHEELS

For the first set of three equations we elect to write the law for motion of the entire system--vehicle plus three wheels:

$$\dot{\bar{H}}^{\text{system}} = \dot{\bar{H}}^{\text{veh.}} + \dot{\bar{H}}^{w} = \bar{M}$$

Details of the derivation are given in Appendix A. The result is:

$$\begin{bmatrix} I_1p^2 + (I_2-I_3)\omega^2 + J\Omega_2\omega & J\Omega_3 p & (I_1-I_2+I_3)\omega p + I_1\dot{\omega} - J\Omega_2 p \\ -J(\Omega_3 p + \Omega_1\omega) & I_2p^2 & J(\Omega_1 p - \Omega_3\omega) \\ -(I_1-I_2+I_3)\omega p - I_3\dot{\omega} + J\Omega_2 p & -J\Omega_1 p & I_3p^2 + (I_2-I_1)\omega^2 + J\Omega_2\omega \end{bmatrix} \bar{\theta}$$

$$+ J\begin{bmatrix} p & 0 & \omega \\ 0 & p & 0 \\ -\omega & 0 & p \end{bmatrix} \bar{\Omega} = \bar{M} - \begin{bmatrix} 0 \\ I_2\dot{\omega} \\ 0 \end{bmatrix} \qquad \text{(A-4)}$$

in which p represents the operation $\frac{d}{dt}$, the components of $\bar{\theta}$ are attitude angles $\theta_1, \theta_2, \theta_3$, the components of $\bar{\Omega}$ are the angular velocities of the three wheels, $\Omega_1, \Omega_2, \Omega_3$, and the components of $\bar{M}$ are the body-axis components of external torque, M_1, M_2, M_3. (J has been lumped with I in the θ terms.)

Often external torques are attitude dependent, as discussed in the preceding section. Typically, aerodynamic torque will have a specific time variation if the vehicle remains always oriented exactly to its reference axes; but if the vehicle deviates, then the torque may vary also as a function of attitude angle θ. In the above equations, therefore, it will be convenient to divide total external torque $\bar{M}$ into a component, $\bar{L}$, which is independent of attitude plus the θ-dependent components, which we assume can be written as linear functions of θ because we shall confine ourselves to very small θ. Then the above equations become:

$$\begin{bmatrix} I_1p^2 + (I_2-I_3)\omega^2 + J\Omega_2\omega + k_{11} & J\Omega_3 p + k_{12} & (I_1-I_2+I_3)\omega p + I_1\dot{\omega} - J\Omega_2 p + k_{13} \\ -J\Omega_3 p - J\Omega_1\omega + k_{21} & I_2p^2 + k_{22} & J\Omega_1 p - J\Omega_3\omega \\ -(I_1-I_2+I_3)\omega p - I_3\dot{\omega} + J\Omega_2 p + k_{31} & -J\Omega_1 p + k_{32} & I_3p^2 + (I_2-I_1)\omega^2 + J\Omega_2\omega + k_{33} \end{bmatrix} \bar{\theta}$$

$$+ J\begin{bmatrix} p & 0 & \omega \\ 0 & p & 0 \\ -\omega & 0 & p \end{bmatrix} \bar{\Omega} = \bar{L} - \begin{bmatrix} 0 \\ I_2\dot{\omega} \\ 0 \end{bmatrix} \qquad (21)$$

Note that the k's can be either positive or negative, and can be time varying. Further, although the k's due to gravity gradient will appear only on the diagonal of the matrix, because 1, 2, 3 are principal axes in Fig. 12, other torques--for example the aerodynamic k's--may not be symmetrical (e.g., because of paddles unsymmetrically deployed): hence the off-diagonal k's in Eq.(21).

In this paper we wish to study some response relations for the system under the simplifying assumptions that the equations are linear with constant coefficients. In some cases these assumptions will be quite accurate. For example, if changes in the wheel speeds are to be small in a given response, then the perturbation technique can be used to linearize the $(J\Omega)\dot{\theta}$ terms. If, further, the orbit is circular, then ω is constant. (If the reference is an inertial one, ω is zero.) Under these conditions Eqs. (21) become linear with constant coefficients.

In other cases the assumption of linearity and/or constant coefficients can serve only as a first approximation, the utility of which must be evaluated in each specific case.

The second set of three dynamic equations is obtained by writing Newton's law for each of the wheels about its spin axis;

$$J(\dot{\Omega}_j + \ddot{\theta}_j) = -T_{c_j}$$

in which $j = 1, 2,$ or 3. (As before, the minus sign occurs because T_c, the torque between the wheel and the vehicle, is considered positive on the vehicle and negative on the wheel.) In most cases of interest, $\ddot{\theta}$ will always be very small (of order J/I) compared to $\dot{\Omega}$, and will therefore be dropped:

$$J\dot{\Omega}_j = -T_{c_j} \tag{22}$$

Control to an Inertial Reference.

Note, first, that Eqs.(21) can be linearized and completely decoupled on the basis of a specific set of assumptions. These are (1) that the reference frame is an inertial one, (2) that the wheel speeds are initially near zero and do not change much during the motions of interest, and (3) that the vehicle has aerodynamic, as well as inertial, symmetry with respect to its principal axes so that cross-torques, k_{ij}, are all zero. These assumptions produce, respectively, the following simplifications in (21): (1) $\omega = 0$, (2) $(J\Omega)\dot{\theta} = 0$, and (3) $k_{ij} = 0$. Then all off-diagonal terms disappear, all on-diagonal terms are linearized, and (21) may be Laplace transformed (if desired) giving:

$$\begin{bmatrix} I_1 s^2 + k_{11} & 0 & 0 \\ 0 & I_2 s^2 + k_{22} & 0 \\ 0 & 0 & I_3 s^2 + k_{33} \end{bmatrix} \bar{\Theta} + Js\bar{\Omega} = \bar{L} \qquad (23)$$

Each of Eqs.(23) is an independent linear equation analogous to Eq.(1a). Correspondingly, Eq.(22) is analogous to (1c). Therefore, all of the results of the section on single axis control apply verbatim to the system described in the preceding paragraph.

Consider next the situation in which we remove special assumptions (2) and (3) above, but retain assumption number (1) (that the reference is an inertial one). In this case, since we are permitting the wheels to have high initial speeds, the terms $(J\Omega)\dot{\theta}$ in (21) will be of first-order importance. However, if the <u>changes</u> in wheel speed during a period of interest are small compared to the initial speed, then the equations can still be linearized by using the perturbation technique. That is, we assume that each wheel speed consists of a constant initial valueplus a small perturbation:

$$\Omega_j = \Omega_{jo} + \Omega_j$$

Then, after Laplace transformation, Eqs.(21) become

$$\begin{bmatrix} (I_1 s^2 + k_{11}) & (h_3 s + k_{12}) & (-h_2 s + k_{13}) \\ (-h_3 s + k_{21}) & (I_2 s^2 + k_{22}) & (h_1 s + k_{23}) \\ (h_2 s + k_{31}) & (-h_1 s + k_{32}) & (I_3 s^2 + k_{33}) \end{bmatrix} \bar{\Theta} + Js\bar{\Omega} = \bar{L} \qquad (24)$$

in which $h = J\Omega_o$. The gyroscopic coupling terms are very much in evidence.

It is shown in Reference (2) that there is considerable advantage in choosing the control equations for a coupled system on the basis of desired performance and, further, of specifying that the responses about the three axes be decoupled from one another. Specifically, let it be required that a disturbance about the principal axis of the vehicle produce a critically-damped response about that axis (as in the single-axis case) and <u>no</u> response about either of the other axes. That is, the response is to be:

$$\begin{bmatrix} I_1\left(\frac{\tau s+1}{\tau}\right)^2 & 0 & 0 \\ 0 & I_2\left(\frac{\tau s+1}{\tau}\right)^2 & 0 \\ 0 & 0 & I_3\left(\frac{\tau s+1}{\tau}\right)^2 \end{bmatrix} \bar{\Theta} = \bar{L} \qquad (25)$$

which has already been plotted in Fig. 5 for $\bar{L}$ an impulse. The relation between τ and θ_{max} is given in more detail in Fig. 6. Equation (24) can now be solved directly for $\bar{\Omega}$, substituting for L from (25):

$$\bar{\Omega} = \frac{1}{Js}\begin{bmatrix} I_1\frac{(2\tau s+1)}{\tau^2} - k_{11} & -(h_3 s + k_{12}) & h_2 s - k_{13} \\ h_3 s - k_{21} & I_2\frac{(2\tau s+1)}{\tau^2} - k_{22} & -(h_1 s + k_{23}) \\ -(h_2 s + k_{31}) & h_1 s - k_{32} & I_3\frac{(2\tau s + 1)}{\tau^2} - k_{33} \end{bmatrix}\bar{\theta} \tag{26}$$

Thus, while the vehicle response to a disturbance is identical with the single-axis case, the wheel-speed response involved is now more complicated. It can be obtained directly in terms of disturbance L by substituting (25) into (26) and solving for Ω. The result is:

$$\bar{\Omega} = \frac{1}{(\tau s+1)^2}\begin{bmatrix} 2\tau s+1-\frac{k_{11}}{I_1}\tau^2 & -\left(\frac{h_3\tau}{I_2}\tau s + \frac{k_{12}}{I_2}\tau^2\right) & \frac{h_2\tau}{I_3}\tau s - \frac{k_{13}}{I_3}\tau^2 \\ \frac{h_3\tau}{I_1}\tau s - \frac{k_{21}}{I_1}\tau^2 & 2\tau s+1 - \frac{k_{22}}{I_2}\tau^2 & -\frac{h_1\tau}{I_3}\tau s + \frac{k_{23}}{I_3}\tau^2 \\ -\left(\frac{h_2\tau}{I_1}\tau s + \frac{k_{31}}{I_1}\tau^2\right) & \frac{h_1\tau}{I_2}\tau s - \frac{k_{32}}{I_2}\tau^2 & 2\tau s+1 - \frac{k_{33}}{I_3}\tau^2 \end{bmatrix}\bar{L} \tag{27}$$

Each of the terms in (27) is of the form:

$$C\,\frac{s + a}{(s + \frac{1}{\tau})^2}\,L(s)$$

If L is an impulse, then the time response corresponding to each term in (27) will resemble, in both form and magnitude, the single-axis response given by Eq.(9) and plotted in Fig. 8, as is evident from a comparison of terms in (27) with (9).

The point indicated by Eqs.(27) is that with high initial wheel speed the wheels will be two or three times as busy and will respond to inputs about all three axes.

The terms $\frac{k\tau^2}{I}$ in (27) represent the relative "stiffness" of the control system and the natural restoring (or destabilizing) torques. In the usual case, the control system will dominate, so that these terms will be small. We therefore drop them at this point. (Their inclusion adds a little complication, but is straightforward, as was discussed in the Single-Axis

Section. The special case of a control system which damps natural motions of a local level system is discussed in the last section of the paper.)

Terms $\frac{h\tau}{I}$ represent the coupling strength, of course, and may be as large as 1 for high-capacity wheels.

The control torques involved in accelerating the wheels as specified in (27) are available from (22):

$$T_c = - Js\Omega$$

in which the components of Ω are given by (27). The form of torque response will therefore be like that given by Eq. (7) and plotted in Fig. 7.

To obtain power consumption we recall that we have selected the case in which change in wheel speed is much smaller than the initial value, Ω_0. The power will therefore be given, closely, by:

$$\bar{P} = - \bar{T}_c \cdot \bar{\Omega}_0$$

That is;

$$\begin{bmatrix} P_1(s) \\ \\ P_2(s) \\ \\ P_3(s) \end{bmatrix} = \begin{bmatrix} \frac{(2\tau s+1)}{(\tau s+1)^2}\Omega_{10} & \left(\frac{h_3\tau}{I_2}\right)\frac{\tau s}{(\tau s+1)^2}\Omega_{10} & -\left(\frac{h_2\tau}{I_3}\right)\frac{\tau s}{(\tau s+1)^2}\Omega_{10} \\ -\left(\frac{h_3\tau}{I_1}\right)\frac{\tau s}{(\tau s+1)^2}\Omega_{20} & \frac{(2\tau s+1)}{(\tau s+1)^2}\Omega_{20} & \left(\frac{h_1\tau}{I_3}\right)\frac{\tau s}{(\tau s+1)^2}\Omega_{20} \\ \left(\frac{h_2\tau}{I_1}\right)\frac{\tau s}{(\tau s+1)^2}\Omega_{30} & -\left(\frac{h_1\tau}{I_2}\right)\frac{\tau s}{(\tau s+1)^2}\Omega_{30} & \frac{(2\tau s+1)}{(\tau s+1)^2}\Omega_{30} \end{bmatrix} \begin{bmatrix} \ell_1 \\ \\ \ell_2 \\ \\ \ell_3 \end{bmatrix} \tag{28}$$

Two forms of time response are obtained from (28), namely:

$$\left(2 - \frac{t}{\tau}\right) e^{-\frac{t}{\tau}}, \text{ and } \left(1 - \frac{t}{\tau}\right) e^{-\frac{t}{\tau}} .$$

These are plotted in Fig. 13 over time periods of interest.

The total energy consumed by the control system during its response to a set of impulses, ℓ_1, ℓ_2, ℓ_3, about the three vehicle axes, is given by

$$E = \int_{t=0}^{\infty} [P_1(t) + P_2(t) + P_3(t)]\, dt$$

From the results of Fig. 13, the total energy can be written explicitly for two cases: (1) energy is not recovered ~~(areas in Fig. 13 are added on an absolute magnitude basis)~~: (only positive areas are considered in Fig. 13; negative areas being obtained by braking):

$$E = \ell_1 \left[\left(1 + \frac{\cancel{2}\,1}{e^2}\right)\Omega_{10} + \frac{\cancel{2}\,1}{e}\left(\frac{h_2\tau}{I_1}\Omega_{30} + \frac{h_3\tau}{I_1}\Omega_{20}\right)\right]$$

$$+ \ell_2 \left[\left(1 + \frac{\cancel{2}\,1}{e^2}\right)\Omega_{20} + \frac{\cancel{2}\,1}{e}\left(\frac{h_3\tau}{I_2}\Omega_{10} + \frac{h_1\tau}{I_2}\Omega_{30}\right)\right] \tag{29a}$$

$$+ \ell_3 \left[\left(1 + \frac{\cancel{2}\,1}{e^2}\right)\Omega_{30} + \frac{\cancel{2}\,1}{e}\left(\frac{h_1\tau}{I_3}\Omega_{20} + \frac{h_2\tau}{I_3}\Omega_{10}\right)\right]$$

(2) energy is recovered (areas in Fig. 13 are added with signs as shown):

$$E = \ell_1\Omega_{10} + \ell_2\Omega_{20} + \ell_3\Omega_{30} \quad ! \tag{29b}$$

The last result suggests the striking possibility that energy consumption may be reduced to the conservative, single-axis level (Eq.(11b)) without storage, by effecting the appropriate transfer of energy from one control wheel directly into another, the proper transfer program to be specified by the computer.

In the absence of energy recovery the total energy, (29a) will be somewhat higher than for the single-axis case.

Response to Sinusoidal Disturbance.

If the time constant of the control system is short compared to the period of a sinusoidal disturbance to which the system is subjected, then--as discussed under single-axis response--the response of the system is greatly simplified because terms $\tau\omega_f$ can be dropped compared to 1 in expressions like (25). Commonly the disturbing frequency is orbital or twice orbital frequency and the system time constant is many times faster, so that this is a good approximation.

If the disturbance is

$$L = L_o \cos \omega_f t \quad ,$$

and if $\tau\omega_f < 1$, then the attitude response (which is single-axis because of the decoupling control) is, from (25):

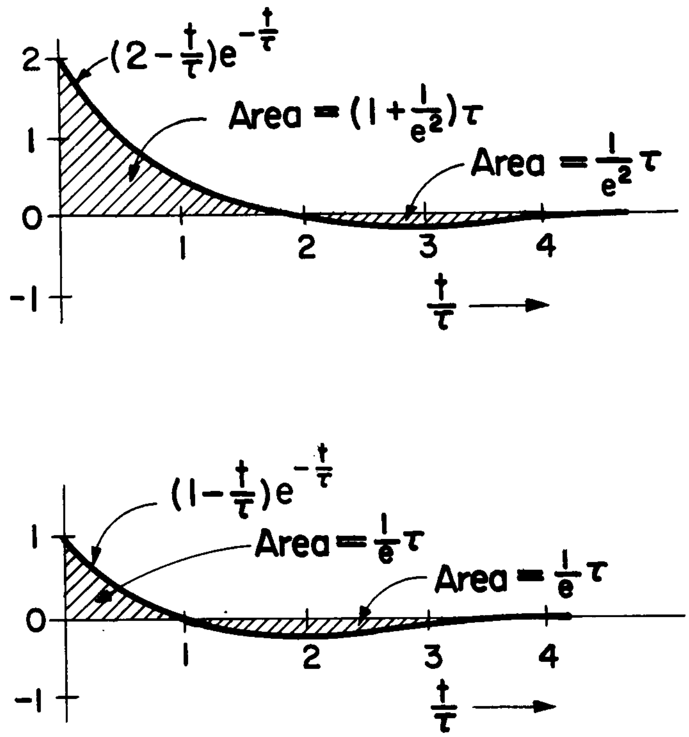


FIG. 13 TIME FUNCTIONS RESULTING FROM EQUATIONS (28).

$$\theta_j = \frac{L_o \tau^2}{I_j} \cos \omega_f t$$

about each axis (which is the same response as (14)).

The wheel speed and control torque--and hence power and energy--also revert to the single-axis form because all coupling terms in (26) contain $\tau\omega_f$*. Thus Ω, T_c, P, and E are given by single-axis relations (16), (15), (17), and (18), respectively.

CONTROL TO A ROTATING REFERENCE

If the reference frame is rotating, at angular velocity ω, then two conditions are necessary to linearize Eqs. (21): (1) $\dot{\theta}$ must be small compared to ω, (2) $\dot{\omega}$ must be small compared, for example, to $\frac{J}{I}\omega\Omega$. If both these conditions are met then (21) can be linearized and Laplace transformed, giving (with $\omega_{mean} = n$):

$$\begin{bmatrix} I_1 s^2 + (I_2 - I_3)n^2 + k_{11} & 0 & (I_1 - I_2 + I_3)ns \\ 0 & I_2 s^2 + k_{22} & 0 \\ -(I_1 - I_2 + I_3)ns & 0 & I_3 s^2 + (I_1 - I_2)n^2 + k_{33} \end{bmatrix} \bar{\theta} + J \begin{bmatrix} s & o & n \\ o & s & o \\ -n & o & s \end{bmatrix} \bar{\Omega} = \bar{L} \tag{30}$$

provided only that the vehicle has symmetry with respect to external torques. (If this is not true, then the Θ matrix will have k_{ij}'s in place of its 0's, and the analysis will be somewhat more complicated, but straightforward.)

In studying the response to sinusoidal disturbance at orbital frequency condition (1), above, reduces to

$$n\theta_{max} \ll n$$

which is met by the restriction to small θ. (θ is deviation from the reference axis system.) For faster transient motions condition (1) may be met only approximately, and must be assessed in each case.

---

*Except for terms k_{ij} (e.g., aerodynamic coupling due to non-symmetry). In the unusual case that these are large enough to cause serious coupling the present analysis must be modified accordingly.

Condition (2) is met exactly, of course, only by a circular orbit. Deviation from condition (2) depends on orbit eccentricity, and must also be evaluated in each case to determine the applicability of linear analysis based on (30). (If deviation is too large then, of course, linear analysis can give only a rough approximation to the behavior, and non-linear techniques or computer solutions are indicated.)

It is convenient to divide systems for controlling a local-level vehicle into two classes: precise control systems, and systems for damping natural motions.

Precise Control System.

If precise, relatively fast control of the vehicle represented by (30) is required then (1) the torques involving n and k in the Θ matrix of (30) will turn out to be small compared with the control torques, and will have negligible effect on the dynamics, but (2) the n terms in the Ω matrix have an important effect involving initial wheel speed.

To show these two points most directly we choose a decoupling control system, such that vehicle motion will be given, again, by:

$$\begin{bmatrix} I_1\left(\frac{\tau s+1}{\tau}\right)^2 & 0 & 0 \\ 0 & I_2\left(\frac{\tau s+1}{\tau}\right)^2 & 0 \\ 0 & 0 & I_3\left(\frac{\tau s+1}{\tau}\right)^2 \end{bmatrix} \overline{\Theta} = \overline{L}$$

The required wheel speed relations are, from (30):

$$Js\Omega_2 = \frac{2\tau s + 1 - \frac{k_{22}}{I_2}\tau^2}{(\tau s + 1)^2} L_2 + J\Omega_2(o) \qquad \text{(pitch)}$$

$$\begin{bmatrix} s & n \\ -n & s \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_3 \end{bmatrix} = \begin{bmatrix} \dfrac{2\tau s+1-\frac{I_2-I_3}{I_1}n^2\tau^2-\frac{k_{11}}{J_1}\tau^2}{(\tau s+1)^2} & \dfrac{-(I_1-I_2+I_3)ns\tau^2}{I_1(\tau s+1)^2} \\ +\dfrac{(I_1-I_2+I_3)ns\tau^2}{I_3(\tau s+1)^2} & \dfrac{2\tau s+1-\frac{I_2-I_1}{I_3}n^2\tau^2-\frac{k_{33}}{I_3}\tau^2}{(\tau s+1)^2} \end{bmatrix} \begin{bmatrix} L_1 \\ L_3 \end{bmatrix} + J \begin{bmatrix} \Omega_1(o) \\ \Omega_2(o) \end{bmatrix}$$

(roll-yaw) (31)

in which pitch and roll-yaw have been decoupled, and initial wheel speeds have been included.

For precise control τ will be very short compared to either $1/\sqrt{k/I}$ or $1/n$, and corresponding terms may be dropped from (31), proving point (1). Then (31) becomes:

$$Js\Omega_2 = \frac{2\tau s+1}{(\tau s+1)^2} L_2 + J\Omega_2(o) \quad ,$$

$$J\begin{bmatrix} s & n \\ -n & s \end{bmatrix}\begin{bmatrix} \Omega_1 \\ \Omega_3 \end{bmatrix} = \frac{2\tau s+1}{(\tau s+1)^2}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} + \begin{bmatrix} \Omega_1(o) \\ \Omega_3(o) \end{bmatrix}$$

Solving for Ω's explicitly:

$$\left.\begin{aligned}
\Omega_2 &= \frac{L_2}{J}\,\frac{(2\tau s+1)}{s(\tau s+1)^2} + \frac{\Omega_2(o)}{s} \qquad , \\
\Omega_1 &= \frac{\frac{L_1}{J}s - \frac{L_3}{J}n\,(2\tau s+1)}{(\tau s+1)^2\,(s^2+n^2)} + \frac{s\Omega_1(o) - n\Omega_3(o)}{(s^2+n^2)} \\
\Omega_2 &= \frac{\frac{L_3}{J}s + \frac{L_1}{J}n(2\tau s+1)}{(\tau s+1)^2\,(s^2+n^2)} + \frac{s\Omega_3(o) + n\Omega_1(o)}{(s^2+n^2)}
\end{aligned}\right\} \tag{32}$$

From Eqs.(32) the pitch wheel motion in orbit behaves just as in the single-axis case (compare with (8)), but the roll and yaw wheels have now also a sinusoidal motion at orbital frequency. This motion will persist--if there is an initial wheel speed--<u>even in the absence of disturbances</u>. This is point (2).

Physically the reason is that the momentum vector stored in the wheels must be maintained <u>fixed in space</u> and, therefore, must be passed from one wheel to the other as the vehicle rotates at angular velocity n. This requires control torque, of course, and therefore power.

The power required is obtained by combining (22) with (32). For no disturbance and $\Omega_2(o) = 0$, for example the amount is

$$P = Jn\,\Omega_1(o)^2\,\frac{\sin 2nt}{2} \tag{33}$$

which may represent a substantial proportion of the total mechanical power required.

The above situation is investigated in greater detail in Reference (3). In particular it is shown there that if decoupling control is not used the momentum exchange must be accompanied by an attitude error which may be very large.

It should be pointed out that the only way to avoid the requirement to accelerate and decellerate in a local-level satellite is to use spinning members which are so mounted that their spin axes are free to remain inertially fixed--as with the reaction sphere.

System for Damping Natural Motions.

From (30) the possibility is evident of using physical restoring torques (e.g., k's) to stabilize a local-vertical satellite. The most commonly used torque is the gravity gradient.

The natural dynamic characteristics of a rigid vehicle in a gravity-gradient field have been described carefully elsewhere (c.f. References (8) (9) and (11).) To summarize briefly, the corresponding values of the k's in (30) are, for a symmetrical vehicle:

$$k_{11} = (I_2 - I_3)\frac{3K_g}{R^3} \qquad \text{(roll)}$$

$$k_{22} = (I_1 - I_3)\frac{3K_g}{R^3} \qquad \text{(pitch)}$$

$$k_{33} = 0 \qquad \text{(yaw)}$$

in which K_g is a gravitational constant. Further, for a circular orbit, $K_g/R^3 = n^2$, so that (30) becomes

$$I_2\left[s^2 + \frac{I_1 - I_3}{I_2} 3n^2\right]\theta_2 + J_d s\Omega_2 = L_2$$

$$\begin{bmatrix} I_1 s^2 + (I_2 - I_3)4n^2 & (I_1 - I_2 + I_3)ns \\ -(I_1 - I_2 + I_3)ns & I_3 s^2 + (I_2 - I_1)n^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_3 \end{bmatrix} + J_d\begin{bmatrix} s & n \\ -n & s \end{bmatrix}\begin{bmatrix} \Omega_1 \\ \Omega_3 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_3 \end{bmatrix} \qquad (34)$$

in which J_d is the inertia of the damping wheel.

Evidently, with the proper relative magnitudes of the I's the vehicle will be neutrally stable due to gravity-gradient alone--i.e., with the wheel speeds zero. That is, it will perform undamped oscillations about all three axes at simple multiples of orbit frequency.

For example the celebrated dumbbell configuration oriented so that $I_2 = I_1$, $I_3 = 0$, will oscillate at frequency $\sqrt{3}\,n$ in pitch and $2n$ in roll. More generally, requirements on the I's are that either $I_2 \geq I_1 \geq I_3$ or part of the region $I_1 \geq I_3 \geq I_2$ for stability. Dynamic characteristics of rigid satellite vehicles are described comprehensively in Reference (11). Note that yaw stability is achieved because of roll-yaw inertial coupling produced by the rotating reference frame. Basically, this is a form of gyrocompassing.

From (34) it is easy to see how light damping of a naturally-stable vehicle can be accomplished. In pitch, for example, we can convert (34) to

$$I_2\left(s^2 + 2\frac{s}{\tau_d} + \omega_p^2\right) \theta_2 = L_2 \tag{35}$$

by letting

$$\Omega_2 = \frac{2I_2}{J\tau_d} \theta_2 \tag{36}$$

(ω_p is an abbreviation for the vehicle natural frequency in pitch, $\sqrt{\frac{I_1 - I_3}{I_2} 3n^2}$, τ_d is the time constant of the damping envelope, and J_d is damping wheel inertia).

From (35) the response to an impulse disturbance, $L_2 = \ell_2 \delta(t)$, will be

$$\theta_2 = \frac{\ell_2}{I_2} \frac{1}{\omega_{pd}} e^{-\frac{t}{\tau_d}} \sin \omega_{pd} t \tag{37}$$

(in which ω_{pd} is the damped natural frequency, $\omega_{pd} = \sqrt{\omega_p^2 - \frac{1}{\tau_d^2}}$). Corresponding wheel speed will be

$$\Omega_2 = \frac{2\ell_2}{J_d \tau_d \omega_{pd}} e^{-\frac{t}{\tau_d}} \sin \omega_{pd} t \tag{38}$$

and control torque:

$$T_{c2} = \frac{2\ell_2}{\tau_d} e^{-\frac{t}{\tau_d}} \frac{\cos(\omega_{pd} t + \psi)}{\cos \psi} \tag{39}$$

$$\psi = \sin^{-1} \zeta$$

for $\zeta < 1$.

(ζ is the damping ratio, $\zeta = \frac{1}{\tau_d \omega_p}$.)

Power is given by the product of (38) and (39):

$$P = \frac{4\ell_2^{\,2}}{J_d \tau_d^{\,2} \omega_{pd}} \; e^{-2\frac{t}{\tau_d}} \left[\frac{\sin 2\omega_{pd} t}{2} - \tan \psi \sin^2 \omega_{pd} t \right] \qquad (40)$$

Comparison of (37), with (4), shows that for the light-damping system attitude excursions will be larger than for precision control by $\frac{1/\tau}{\omega_{pd}}$.

Comparison of (38) with (8) shows that, for the same control-wheel speeds, the damping wheel may be made smaller by:

$$\frac{J_d}{J} = \frac{2}{\tau_d \omega_{pd}} = 2\zeta \quad .$$

In this case comparison of (39) and (40) with (6) and (9) shows that control torque and power will both be lower, by τ_d / τ, for the damping system:

$$\frac{P_{damper}}{P_{precision\ controller}} \approx \frac{\tau_{precision\ controller}}{\tau_{damper}} \qquad (41)$$

Equation (41) is a convenient relation for preliminary design studies.

In roll-yaw the decoupling principle is again useful, and we can specify response by

$$\begin{bmatrix} I_1(s^2 + \frac{2}{\tau_d} s + \omega_r^{\,2}) & 0 \\ 0 & I_3(s^2 + \frac{2}{\tau_d} s + \omega_y^{\,2}) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_3 \end{bmatrix} \qquad (42)$$

in which $\omega_r^{\,2} = \frac{I_2 - I_3}{I_1} 4n^2$, $\omega_y^{\,2} = \frac{I_2 - I_1}{I_3} n^2$. (The same τ_d is chosen in all three axes in this example.)

Then the wheel program required will be, from (34):

$$J_d \begin{bmatrix} s & n \\ -n & s \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_3 \end{bmatrix} = \begin{bmatrix} I_1 \frac{2}{\tau_d} & -(I_1-I_2+I_3)n \\ (I_1-I_2+I_3)n & I_3 \frac{2}{\tau_d} \end{bmatrix} s \begin{bmatrix} \theta_1 \\ \theta_3 \end{bmatrix} \qquad (43)$$

From (42) the attitude response in roll and yaw will be the same as in pitch, (37).

The wheel motion (and therefore, also, torque and power) in roll-yaw will be somewhat more involved than in pitch because of the off-diagonal terms in (43). Substituting for θ from (42) the response of wheel 1 to an impulsive torque (about an axis in the roll-yaw plane) is given by:

$$\Omega_1 = \frac{2\ell_1}{J_d \tau_d} s \left[\frac{s - \left(\frac{I_1-I_2+I_3}{2I_1}\right) n^2 \tau_d}{(s^2 + \frac{2}{\tau_d} s + \omega_r^2)(s^2+n^2)} \right] + \left(\frac{I_1-I_2+I_3}{I_3} \right) n \ell_3 s \left[\frac{s - \frac{2}{\tau_d}\left(\frac{I_3}{I_1-I_2+I_3}\right)}{(s^2 + \frac{2}{\tau_d} s + \omega_y^2)(s^2+n^2)} \right]$$

The expression for Ω_3 is similar. The first denominator quadratic leads to a time response similar to (38). But, in addition, the second denominator quadratic leads to a component of wheel speed which is sinusoidal at orbital frequency, just as there was in (32). Again, this is necessary to keep the wheel momentum vector fixed in space when there is no disturbance, and cannot be avoided in control to a local-vertical reference with the system of Fig. 12. (Note that when $n=0$, the above equation becomes just like (36).) Moreover, for the present system in which ω_r and ω_y are slow, of order n, the peak power required to accelerate and decelerate the wheels during the orbits following a single impulse will be as large as that required to damp the transient attitude motions produced by the impulse.

Sinusoidal Response.

For a damper system the sinusoidal response relations are more involved than for a tighter control system simply because the forcing frequency is no longer much lower than the natural frequency of the system, but, on the contrary, may be somewhere between $\frac{1}{2}$ the natural frequency (for disturbance of a dumbbell in roll at orbital frequency) to several times the natural frequency (e.g., for disturbance of a more nearly iso-inertial vehicle.)

For a gravity-stabilized vehicle it is assumed that the natural frequency of the vehicle--e.g., $\sqrt{3 \frac{I_2-I_3}{I_1} n^2}$ in pitch--will be at least

a sizeable fraction of orbital frequency. Otherwise gravity stabilization would be too weak to be useful in the presence of other torques.

To illustrate the relations, the frequency response of a vehicle in pitch is shown in Fig. 14. Accompanying plots show wheel speed and torque response. It is seen to be advantageous to avoid resonance, if possible.

Eccentric Orbit.

It must be emphasized again that the present analysis assumes a near-circular orbit. When the orbit is eccentric the gravity gradient may actually be destabilizing, so that more positive control is required and the technique of merely damping natural motions cannot be used. (See Reference 8.)

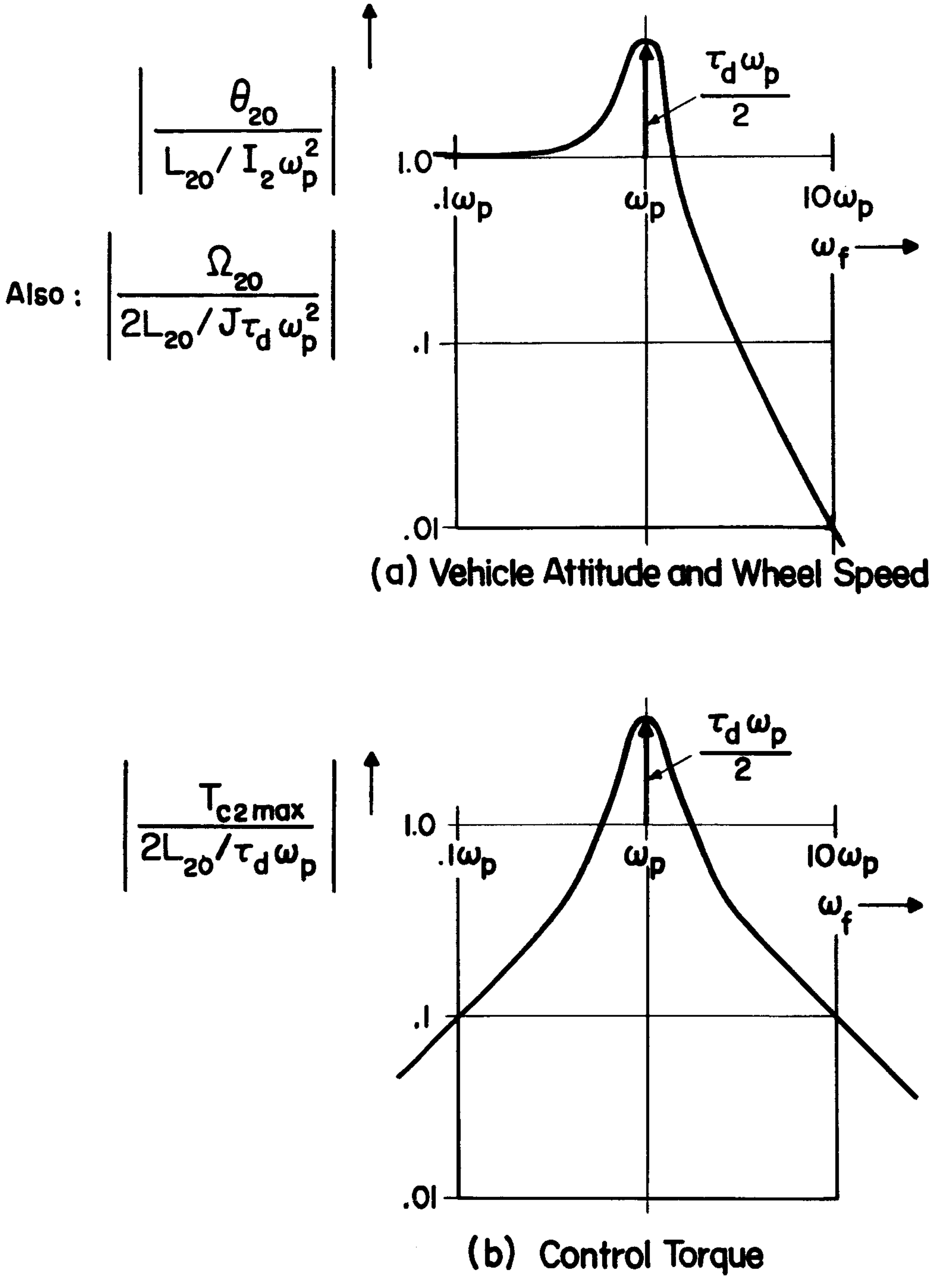


FIG. 14 RESPONSE OF DAMPING SYSTEM TO SINUSOIDAL DISTURBANCE IN PITCH.

SUMMARY

In the common attitude control system which utilizes a combination of momentum-expelling and momentum-storage mechanisms the latter draws the job of furnishing control precision and speed of response with minimum power.

Design of a reaction wheel system for required response is aided by basic response relations for initial error and impulsive and sinusoidal disturbances, as illustrated in Figs. 4 through 10 and Eqs. (13) through (18) of the present paper. Specifically, controlled-system ~~time~~ constant can be chosen from allowable attitude escursion in Fig. 6, for an impulsive, or Eq. (14) for a sinusoidal disturbance; or from allowable initial recovery time in Fig. 4. Torquer capacity and wheel size can be determined from Figs. 7 and 8, for an impulsive disturbance,* or from Eqs. (15) and (16) for a sinusoidal one. Peak power and energy are given by Fig. 9 and Eq. (11) for an impulse, or by Eq. (17) and (18) for a sinusoidal disturbance.

While the above relations are derived for the single axis case, their applicability to three-axis design is established. For precision control to an inertial reference the postulation of decoupling control--which offers several advantages--results in vehicle response which is identical with the single-axis case, but control wheel activity may be greater by a factor of 2 or 3, due to gyroscopic coupling, if the initial wheel speeds are high. Even so, if a technique can be devised for recovering mechanical energy, then energy consumption may be reduced to the single-axis level. The possibility of mechanical energy conservation is discussed in Appendix B.

For precision control to a rotating reference the single-axis vehicle response may still be achieved, but control activity is now complicated by the continuous angular velocity of the vehicle. In particular, spin momentum must be transferred back and forth between the roll and yaw wheels--at significant expense in power--to keep their momentum vector fixed in space. An accompanying attitude error can be avoided only by using decoupling control.

A control system which merely damps the natural motions of a local-vertical satellite can be used if precision control is not required and if the satellite has a stable configuration and is in a nearly-circular orbit. Then the response of the vehicle will be larger and slower, roll and yaw motions being generally coupled, but smaller wheels can be used (Eq.(39)), and power consumption will be lower by the ratio of damping time, in the damper system, to the control time constant of the precision system.

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* Only mechanical power, required by an idealized torquer, is calculated.

APPENDIX A: EQUATIONS OF MOTION FOR SYSTEM OF FIG. 12

Refer to the assumptions stated under "THREE-AXIS CONTROL".

The angular velocity of the vehicle-fixed coordinate system $\overline{1},\overline{2},\overline{3}$, is

$$\overline{\Omega}^{veh} = \overline{\Omega}^{veh/ref} + \overline{\Omega}^{ref} \tag{A-1}$$

For the assumptions given, that the θ's are small, and that the reference system rotates about $\overline{2}_r$ only, (A-1) can be written:

$$\begin{aligned}\overline{\Omega}^{veh} &\approx \dot{\overline{\theta}} + \overline{2}_r\,\omega \\ &\approx \overline{1}(\dot{\theta} + \omega\,\theta_3) + \overline{2}(\dot{\theta}_2 + \omega) + \overline{3}(\dot{\theta}_3 - \omega\,\theta_1)\end{aligned} \tag{A-2}$$

That is, products of Euler angles and their derivatives are neglected.

The angular momentum of the vehicle is given by:

$$\overline{H}^{veh} = [I]\,\overline{\Omega}^{veh} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \overline{\Omega}^{veh}$$

and the angular momentum of the wheels by:

$$\Sigma\,\overline{H}^{w} = J\,[\overline{\Omega}^{veh} + \overline{\Omega}]$$

in which $\overline{\Omega}$ is defined (as in the text) as a vector made up of the <u>wheel spin</u> velocities:

$$\overline{\Omega} \triangleq \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

Then, with $\overline{H}^{syst} = \overline{H}^{veh} + \Sigma\,\overline{H}^{w}$, we can write:

$$\dot{\overline{H}}^{syst} = \overset{\circ}{\overline{H}}{}^{syst} + \overline{\Omega}^{veh} \times \overline{H}^{syst} = \overline{M} \;* \tag{A-3}$$

---

* $\overset{\circ}{\overline{H}}$ means the derivative of $\overline{H}$ as seen by an observer on the vehicle---i.e., the derivative of $\overline{H}$ with $\dot{\overline{1}} = \dot{\overline{2}} = \dot{\overline{3}} = 0$.

with respect to vehicle coordinate system $\overline{1},\overline{2},\overline{3}$. If, again, one drops the small products of Euler angles and their rates, and also $\frac{J}{I}$ compared to 1, the result is:

$$\begin{bmatrix} I_1p^2+(I_2-I_3)\omega^2+J\Omega_2\omega & J\Omega_3p & (I_1-I_2+I_3)\omega p-J\Omega_2p+I_1\dot{\omega} \\ -J(\Omega_3p+\Omega_1\omega) & I_2p^2 & J(\Omega_1p-\Omega_3\omega) \\ -(I_1-I_2+I_3)\omega p-I_3\dot{\omega}+J\Omega_2p & -J\Omega_1p & I_3p^2+(I_2-I_1)\omega^2+J\Omega_2\omega \end{bmatrix}\overline{\theta}+J\begin{bmatrix} p & 0 & \omega \\ 0 & p & 0 \\ -\omega & 0 & p \end{bmatrix}\overline{\Omega} = \overline{M} - \begin{bmatrix} 0 \\ I_2\dot{\omega} \\ 0 \end{bmatrix} \tag{A-4}$$

in which $p \triangleq \frac{d}{dt}$.

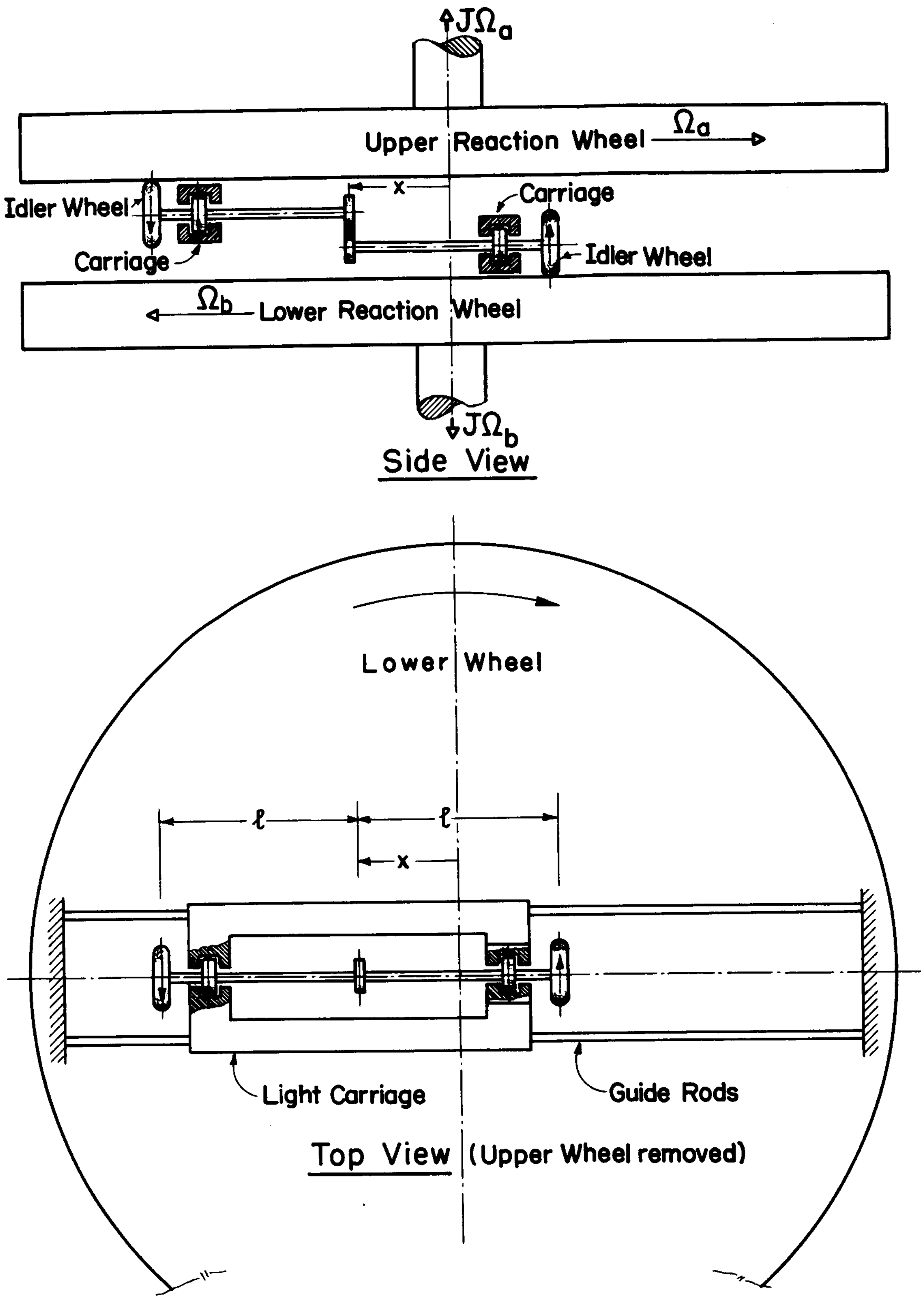

FIG. B A WHEEL SYSTEM WHOSE MOMENTUM CAN BE CONTROLLED WITH (IN THE LIMIT) ZERO ENERGY LOSS

APPENDIX B

THE FEASIBILITY OF CONTROLLING MOMENTUM WITHOUT ENERGY CONSUMPTION

The purpose of this appendix is simply to show that, in principle, the net momentum of a flywheel system can be varied as desired over an indefinite period of time with loss in energy which, in the limit, may be zero. To this end an all-mechanical, energy-conservative system is offered, for demonstration only, in Fig. B.

Two coaxial reaction wheels are arranged with an idler-wheel drive between them. The idler is positioned by a carriage, as shown in the top view.

If the carriage position, x, is zero, then the two wheels run at equal speed, in opposite directions, and the net momentum of the pair is zero. Let this initial wheel speed be $\Omega(o)$. When x has a non-zero value the ratio of the reaction wheel speeds is given by the geometry, and the net momentum of the system can have any value between (in the limit) plus and minus $\sqrt{2}\, J\Omega(o)$, depending on x.

For example, for the position shown, $x = \ell/2$, the lower wheel must be running three times as fast as the upper wheel, and the net momentum of the system is $2J\Omega_a$, directed downward. Moreover, if the system was moved from $x = 0$ to $x = \ell/2$ without energy loss, then Ω_a can be shown to be $.45\ \Omega(o)$ (and $\Omega_b = 1.34\ \Omega(o)$), so that the net momentum of the system is $.89\ J\Omega(o)$.

When a change in x is made the necessary accompanying transfer of momentum, from wheels to vehicle, takes place, of course, via the torque impulse the idlers exert on the carriage (and thence on the vehicle) about the reaction wheel axis. (Unfortunately, the system shown also exerts a small torque impulse about an axis parallel to the idler axes. This must be eliminated, by design, or compensated by the control computer, for three-axis operation.)

To show that operation with negligible energy loss is reasonable we consider, first, that there is viscous friction between the idler and the reaction wheels. Next we let the friction become high and note the limiting values of momentum change and energy change. Finally we consider the energy necessary to position the idlers.

If the idlers in Fig. B are assumed to be massless, and if the friction between idlers and reaction wheels is considered viscous, then the response of the system to a sudden change* in x, from x_1 to x_2, turns out to be given by:

*The sudden-change response is pursued because then x, which appears as a multiplier in the differential equations, is constant.

$$\Omega_a = \Omega_a(o) \frac{s + \frac{(1 - x_2/\ell)}{(1 - x_1/\ell)} (1 + \frac{x_1}{\ell} \frac{x_2}{\ell})\sigma}{s[s + (1 + x_2^2/\ell^2)\sigma}$$

in which σ is the reciprocal of the time constant of one reaction wheel skidding against a clamped idler located at radius ℓ.

If a large change in x is made suddenly from x_1 to x_2, then considerable energy is lost to friction. But if the change is made gradually, over a time which is long compared to the time constant of the idler system, $1/\sigma(1 + x_2^2/\ell^2)$, then the energy lost goes to zero. That is, by making the momentum change gradually it can be made reversibly, in the limit.

But since σ is proportional to viscosity the speed with which momentum can be changed, with negligible energy loss, can be made as fast as desired by increasing viscosity as needed.

We must consider also energy required to move the idlers. Here a possible refinement for reducing skidding is to make the idler wheels steerable, like the wheels of a car. Then a tiny force can steer the wheels rapidly to any desired position with energy loss which, again, can be made very small.

The system presented in this appendix is not operationally optimum for several reasons. (For example, only $1/2\sqrt{2}$ times the total momentum capacity of the pair of wheels is utilized.) But the point is made, it is believed, that it is quite feasible to operate reaction wheel systems satisfactorily for long periods with the torquer system turned off.

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